Understanding the Number Line: Conception and Practice

by

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Declaration

I declare that this thesis has not been submitted for a degree at another university neither the work contained within it has been published elsewhere nor has it been used before, except where specified here.

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Abstract

This study investigates the relationship between teacher's presentation and children's understanding of the number line within an English primary school that follows the curricular guidance presented within the National Numeracy Strategy (DfEE, 1999a).

It was the initial intention of this study to trace the understanding and use of the number line in school by trainee teachers during the final stages of their training. However, after some extensive study with these trainees, for logistic purposes the main aspect of the study was carried out within an English primary school.

Following an exploratory study, which guided the development of a questionnaire, the preparation of a pilot study, and the initial investigation with the trainee teachers, the study was re-conceptualised to consider the way in which teachers within each year group of a primary school used the number line and the ways in which their children conceptualized and interpreted it.

Using a mixed methodology, the theoretical framework of the study draws upon methods associated with case study, action research and ethnography and involved the use of questionnaires, teacher observations and interviews with selected children. Analysis of the questionnaire data is mainly through the use of descriptive statistics that lead to discussion on children embodiments of the number line, their interpretations of what it is and their accuracy in estimating magnitudes. Verbal data obtained from observation and interview is considered through a phenomenographic approach, which adds support to a deeper understanding of the statistical element, whilst the whole is associated with the guidance contained within the National Numeracy Strategy.

The results of the study suggest that conceptualising the number line as a continuous rather than discrete representation of the number system that evolves from the notion of a repeated unit was less important than carrying out actions on the number line. It is suggested that this emphasis caused ambiguity in the way teachers referred to the number line and restricted understanding amongst the children that focused upon the ordering of numbers and the actions that could be associated with this ordering. The results also suggest that children's conceptions of magnitude on a segmented 0 to 100 number line neither meet objectives specified within the National Numeracy Strategy nor confirm hypothesised models that suggest a linear or logarithmic pattern of accuracy.

The outcome of the study suggest that though the teachers were conscientious in their attempts to meet guidance specified within the National Numeracy Strategy, their interpretation of this guidance invokes a demonstrational rather than developmental presentation of this representation of the number system. The number line is seen to be a tool but its use as a tool becomes limited because teachers, and consequently children, display little if any awareness of its underlying structure and its strength as a representation of the number system.

Chapter 1: Introduction

1.1 Focus of the Study

The rationale for this study arose from the researcher's MSc dissertation (Doritou, 2001), which considered children's ability in estimating and rounding numbers. That study was informed by the legal requirements of the English National Curriculum (DfEE, 1999b) and the associated curricula guidance presented within the National Numeracy Strategy (DfEE, 1999a), non-statutory guidance designed to support the teaching and learning of a secure foundation in numeracy.

Though the number line features within the UK curriculum (QCA, 1999; QCA, 2000; DfEE, 1999a) the results of the MSc suggested that there are weakness in children's understanding of its nature. It is this observation that leads to the central issue guiding this study:

> *How is the number line used within the English primary school and how is it understood by children?*

It is the purpose of this chapter to provide an essence of the considerations that will provide the background to answering this questions and an outline of the structure of the study through which it is answered.

Herbst (1997) identified the number line as a metaphor of the number system and defines it as the consecutive translation of a specified segment U as a unit, from zero. U itself can be partitioned in an infinite number of ways (i.e. fractions of U). It is this 'density' that provides the rationale for Herbst's suggestion that the number line can be seen as a representation of the number system. Such a perception suggests that the number line is a sophisticated representation, although the concept that real numbers correspond to linear magnitudes, implying an underlying continuity, has existed since ancient Greek times (Bourbaki, 1984; p. 121).

Herbst also added that the number line may be seen as a "solving and justifying" tool that can help in obtaining the solution to a problem on the line, as well as providing the justification of the way of thinking and consequently to the way in which an answer was obtained.

It is these three themes, the number line as a representation for the number system, as a pedagogical tool and an aid in thinking that guide the issues presented within this study and consequently its central concern which is to investigate the way teaching and learning supports the recognition and the compatibility of these features.

1.2 Understanding and Using the Number Line: The Introductory Context

1.2.1. The Number Line as a Representation of the Number System

To use a metaphor to support thinking there would seem to be the need to know and understand the nature of the "helping tool" so that it can be used flexibly. Cobb, Yakel & Wood (1992) stress that the teacher may interpret mathematical meanings or the use of a material, in different ways to those of learners. A consequence of this is the "learning paradox", where the teacher is not in a position to realise that the pupil sees things from a different perspective and might not be in a position to receive a concept in the same way as the teacher understands it.

This is an issue that seems to be highlighted when we wish to consider the number line. Herbst (1997) provides an indication of the way in which a number line may be formed:

> one marks a point 0 and chooses a segment u as a unit. The segment is translated consecutively from 0. To each point of division one matches sequentially a natural number. (Herbst, 1997, p. 36)

Thus the number line is formed by selecting an interval and then through successive translations of that interval. It is therefore relatively easy to see how the natural numbers (1, 2, 3,...) can be identified by marking the end points of each successive segment starting from the first end point (zero). But of course we can also go the other way and now each successive end point can be marked using negative numbers so that now we have a representation of the integers (..., -2, -1, 0, 1, 2,..). Since we can also partition each segment, we can identify points on the line to position the rational numbers (p/q, where p and q are integers). The irrationals are more problematic. Theoretically, at least we place them on the line but we are limited to a finite degree of accuracy and therefore frequently identify their approximate position relative to two known rationals.

An issue for this study is the perception that teachers and children have of the possibilities available when considering whether or not a labelled number line segment is complete — to what extend is the underlying continuity of the number line recognised?

1.2.2 The Number Line as a Pedagogical Tool

Recognition that the number line possesses certain underlying features has led to a variety of recommended models (Freudenthal, 1973; Anghileri, 2000), suggested modifications (Klein, Beishuizen & Treffers, 1998) and applied uses in, for example, the teaching of whole number operations (Beishuizen, 1997) and fractions (Behr, Harel, Post & Lesh, 1992). Curriculum guidance for teachers within England has indicated that the number line is a "key classroom resource" (DfEE, 1998a, p. 23) that has been found to help children master numeracy skills. It is therefore no surprise that as a representation it occurs frequently within the National Numeracy Strategy as a mechanism to support children's calculation strategies in addition and subtraction. There are recurring references to ether its use or the use of a number track within each year group of the primary school. Within the reception year and all other years, this use is associated with ordering and positioning either the natural numbers or, in later years, integers or fractions. From Year 2 objectives also focus on recording estimates of magnitudes on a number line and finding the difference between that estimate and the actual number. (Perhaps this emphasis was derived from the earlier observation (QCA, 1998) that all but 18% of children had difficult responding to a question that asked them to indicate the magnitude of a series of numbers each number represented by an arrow.) In all years there are references to its use in the development of skill in addition and fractions whilst from Year 4 it plays an additional part in children's reconstruction of their number knowledge to include fractions, decimals and negative numbers.

The general picture that emerges from the use of the number line as a pedagogic tool within the National Numeracy Strategy is one which emphasises procedural applications supported through the use of the number line without any emphasis on its conceptual structure. For instance, the number line is seen as a mechanism for showing how forwards and backwards counting works but there is a sense of ambiguity between its use and the use of the number track. Indeed, differences between the two are indicated by differences associated between labelling spaces and labelling points but there is no reference to what the points actually mean. There is an implicit assumption that the conceptual differences between the number track and the number line are recognized by teachers, applied in practice and understood by children.

Within this study, we will consider the use of the number line as a pedagogic tool and consider the ways in which it is considered within the classroom. This leads to a consideration of the ways teachers talk about it, represent it and use it and the consequent meanings that are established by the children.

1.2.3 The Number Line as an Aid to Thinking

The number line concept is accompanied by interpretation difficulties (Cannon, 1992; Streefland & Heuvel-Panhuizen, 1992). For example, in using it for counting children may count the marks and ignore the intervals in between (Carr & Katterns, 1984; Baturo & Cooper, 1999). They may apply whole number knowledge when they order fractions on the number line (Hartnett & Gelman, 1998) or have difficulties in identifying the whole when fractions are involved on the line (Merenluoto, 2003).

An interesting feature that emerges from any discussion on perceptual salience is the emerging use of an open line to support addition and subtraction approaches. More recently this has been referred to as the Empty Number Line (ENL), essentially a geometric line with neither numbers marked on it nor marks for the intervals, but it has the underlying structure that preserves number order. This originated in Holland at the start of the 1990's where its use has proved to be a success (Klein, Beishuizen & Treffers, 1998). The ENL is suggested to support mental calculation strategies (Beishuizen, 1997; Harries & Spooner, 2000). Within the National Numeracy Strategy, the first, although implicit, reference to the use of such a line is within the specified outcomes for Year 3 indicating that children should use informal pencil and paper jotting to record the addition of two digit numbers (DfEE, 1998, p. 43).

Within this study, we will examine the extent to which the number line is used by children to support their thinking in the field of elementary arithmetic and within the contexts identified within the National Numeracy Strategy.

1.2.4 The Number Line as a Representation for Teaching and Learning

Though the notion of number line may be seen as a representation of the more abstract notion of the number system, within classrooms it usually takes on a concrete form as an instructional representation. Lesh, Post & Behr (1987) identify the number line as a representation that is a manipulable model that possesses in built relationships and operations that may fit every day situations and form the basis for internalised images. Such a model can act as a mediator of mathematical ideas between the teacher and the student. However, Foster (2001) has indicated that representation within the classroom may be used in two ways, either in 'demonstration mode' or in 'developmental mode'. He sees that in the latter children use the representation freely and creatively according to their personal ideas and understanding in order to make sense of it. It is interesting to note that the National Numeracy Strategy emphasises the former in its introduction (DfEE, 1998a, p. 12) but does not mention the latter although there is a need for learners to know and understand the nature of the model they are using.

The general picture that emerges in the context of representations is that although pedagogic representations can help considerably in the teaching and learning of mathematics, they can also be problematic in learning.

> If embodiments are not meaningful, then even though they are "concrete" they turn out to be as "remote" as the object itself. (Janvier 1987, p. 102)

Children may abstract different meanings from what seems to be a "common" representation within a class and later construct concepts that are only meaningful to the individual (Fuson, Wearne, Hiebert, Murray, Human, Olivier, Carpenter & Fennema, 1997; Dufour-Janvier, Bednarz & Belanger, 1987; Bills, 2001). Through their interpretation of children's conceptions of symbolic and verbal representations within elementary arithmetic Pitta & Gray (1997) distinguished between two different kinds of conception, those that were explained in a descriptive way that focus upon specific and/or episodic features of the representation, the latter frequently associated with actions with the representation, and those that are relational in the sense that they are generic and proceptual manifestations of the intrinsic qualities of the representation that enables the learner to make links with other units of knowledge. The former appears to be a feature of the interpretation of lower achieving children whilst the latter is more common amongst high achievers.

Trying to homogenize learners by using a representation that seems appropriate and is meaningful to an adult might not always give support in the learning of a particular mathematical concept (Cobb, Yackel & Wood, 1992). From a constructivist's perspective, we can see that every child has their individual mental constructions on which they may base their abstractions and consequently their creation (or otherwise) of new mathematical concepts. Though children may learn how to use a particular representation, the mathematics the representation represents may not be developed (Dufour-Janvier, Bednarz & Belanger, 1987; Kilpatrick, 1987).

In its examination of issues associated with understanding the nature of the number line and its use in teaching and learning, the study will consider the understanding that the children posses of the number line and in particular their ability to estimate magnitudes on a number line segment.

1.3 The Study

The aim of this study is to gain access to the individual's cognitive processes and discover whether or not the use of a particular representation, the number line, contributes to the construction of "number knowledge[1]".

The main research question is formulated as:

> *How is the number line used and understood within the primary mathematics classroom?*

In association with the discussion above this issue leads to two operational questions that will be considered within the study:

> *What is the primary school teachers' perception of the number line and how do they use it as a pedagogic tool?*
>
> *What is the understanding that primary school children have of the number line and what sense do they have of its underlying features?*

To respond to these questions an exploratory study was carried out to establish the way in which a range of individuals, primary school children, teacher trainees and graduate students perceived the number line. This exploratory study, relatively informal in nature, was essentially a mechanism to consider the issues associated with understanding and interpretation of the number line and from its analysis a pilot study was designed that translated these issues into a more formal means of investigation with children within an English primary school.

The contributions of these two studies, together with a review of relevant literature, guided the development of the main study and suggested that several research instruments were available for use within the study; test, questionnaire, interview and

[1] Here "number knowledge" denotes conceptual knowledge and understanding of the number system

observation. It was finally decided that the main study would draw upon several methodological approaches and consequently the study has a mixed approach that draws upon both qualitative and quantitative data.

The main study is essentially a case study and considers the actions and beliefs of teachers and children in relationship to their conceptions of the number line. Although the original perception of the study was to focus on trainee teachers and, through the use of their teaching documentation, in its initial formulation it had elements associated with action research. However, after going a considerable way towards identifying the sample of teachers and establishing a research relationship with selected individuals, practical difficulties meant that the original study had to be considerably modified. In its final form the study presents a case study of the ways in which teachers and children in a particular school consider the number line.

In the sense that the National Numeracy Strategy is assumed to evoke "wise practice" in its recommendations of the way in which lessons should be structured and in the organisation and presentation of the content there remains an element of action research associated with the study but this does not provide the methodological background that was originally intended. The study does not set out to test a theoretical assumption but attempts to examine the ways through which individuals interpret and use a particular instrument. Since much of the data will be drawn from articulations of individuals' perceptions of the number line phenomenon, data analysis of their perceptions of this phenomenon will make use of the notion of phenomenography (Neuman, 1987, 1999).

1.4 An Outline of the Study

Within the second chapter of the study an overview of theories associated with the development of mathematical thinking is presented (§2.2). It is suggested that there are two distinct types of knowledge (Ryle, 1949; Hiebert & Lefevre, 1986) and understanding (Skemp, 1976). One kind associated with the application of actions and the other with the reasoning behind these actions in addition to their application. More recent theories focusing on the development of increasing degrees of sophistication in the ways that individuals think about mathematical actions and mathematical concepts

(Dubinsky, 1991; Sfard, 1992) are considered whilst the potential divergence that occurs from an emphasis on the former (Gray & Tall, 1994) places a perspective on the advantages of flexibility in thinking. The issue of pedagogic representations is discussed in §2.3 whist a particular focus is given to the number line as a representation within §2.4. Though its use is common this use is not always accompanied by success. Perceptual problems such as those related to its relationship with the measure concept (Streefland & Heuvel-Panhuizen, 1992) and conceptual problems associated with its extension from whole numbers to fractions (Davis, Alston & Maher, 1991) may cause confusion for learners.

The ways in which the National Curriculum and the National Numeracy Strategy advocate the use of the number line is considered within §2.5.

Chapter 3 considers the outcomes of the exploratory study (§3.2) and the pilot study (§3.3). Developed to gain a clearer perception of the way a variety of individuals understand the number line, the exploratory study explored a framework to support resolution of the research questions whilst the 'pilot' study was developed to consider the way that English pupils would react to issues emerging from the 'exploratory' study. Consequently, the pilot study is an integral part of the development of a suitable methodology. Both it and the exploratory study not only contribute to the main study from a data collection perspective but also from the perspective of data analysis.

From these two studies, the areas significant to the development of the main study were identified. Thus within Chapter 4 we see discussion associated with the method used in the main study and the development of the main instruments through which the research questions were answered. §4.2 presents a discussion of the theoretical background and the design options that includes deeper discussion on the development of the mixed methodology and the phenomenographic orientation associated with the study. Within §4.3 there is a detailed consideration of the data collection approaches and the research instruments. The nature of the samples are considered within §4.4 whilst data analysis is the focus of §4.5. Ethical issues and issues of reliability and validity are considered within the remaining two sections.

Chapter 5 considers teachers' understanding of the number line. Though the teacher trainees were not followed within school, their contribution is felt worthy of inclusion since it has a strong relationship with the perceptions of the teachers in the selected school and partly because the greater proportion of their perceptions also seems to mirror those of the children within school. Both trainees' and practicing teachers' understanding of the number line focuses on describing a particular line or on the line as a tool.

Chapter 6 focuses on teacher indications of the way number lines are identified and used during their lesson and upon children's practice and interpretations of these meanings. Since the dominant theme was the use of the number line as a tool, the chapter is subdivided into sections that are associated with the number line and its use in developing knowledge of whole number (§6.2), fractions (§6.3) and decimals (§6.4). Within each section, the data is presented in such a way that it draws upon evidence from the observation of teachers and interviews with selected children from within each year within the selected primary school.

The children's conceptions of the number line are considered in more detail within Chapter 7. The chapter presents the result of the analysis of a questionnaire presented to all of the children within each of the observed classes apart from Year 1. The Chapter considers children's embodiment of the number line (§7.2), their conceptions of what a number line is (§7.3) and the accuracy within which they can estimate magnitudes (§7.4) and identify nominated positions (§7.5).

The discussion and the conclusions are presented within Chapter 8. The relationship between the guideline that is the National Numeracy Strategy and teachers' actual delivery and understanding of the number line is considered within §8.2. The National Numeracy Strategy is seen to make a significant contribution towards teacher's pedagogical content knowledge. It is a contribution which emphasises the use of the number line as a tool but makes little reference to it as an abstract representation of the number system. Consequently, children's conceptions of the number line are articulated in terms of description and action with very limited conceptual reference. Section 8.3 returns to discuss the outcome of the results from the children's attempts to estimate magnitudes on a number line segment. Drawing upon research that only became available in the latter stage of this study, the section considers the distribution of the children's estimates and argues that a model of distribution does not conform to published theoretical models. The discussion suggests that children make use of 'perceptual anchors' formed from the extremes and middle of the segment and these in turn have an effect on the direction of the errors. Section 8.4 presents reflection on the method used within the study whilst section 8.5 considers additional research that may arise out of the study. A final reflection is presented within Section 8.6

Although the National Numeracy Strategy recommends that each classroom should possess a 'large, long number line for teaching purposes' (DfEE, 1998a. p. 29), there is no indication that its use as a tool should be accompanied by conceptual understanding of its structure. In teaching towards the outcomes specified within the National Numeracy Strategy, the teachers draw upon the guidelines presented within the Strategy to display pedagogical content knowledge of the number line that focuses on its use as a tool with little underlying development of what the number line is, how it is structured and its significance as an abstract representation of the number system. The greater majority of children, who emphasise the perceptual characteristics of a specific number line or its use in a variety of arithmetical operations have, in turn, very little perception of its meaning and use in expanding their notions of the number system. These conclusions lead to the thesis of this study:

> *The primary use of the number line as a tool can undermine its strengths as a metaphor of the number system and may disadvantage children in their continued reconstruction of knowledge of that system.*

It is a corollary of this thesis that use of the number line as a representation within the primary school should be withheld until such time as children understand its conceptual structure.

Chapter 2: Literature Review

2.1 Introduction

This study considers the way in which the number line is used within a primary school and the way it is understood and used by children within the latter part of Key Stage 1 and throughout Key Stage 2. Four main themes are central to the study. Within §2.2 we consider theories that can contribute towards our understanding of the development of mathematical knowledge. Piaget and his influence on the work of Sfard (1991), Dubinsky (1991) and Gray & Tall (1994) provide a sense of the way in which thinking associated with mathematical actions, sourced from within the environment or from representations that model that environment, has the potential to rise to a greater level of sophistication through a reconstruction of processes that may be variously identified as interiorisation, reification or encapsulation.

The transformation of such processes into concepts indicates a shift of attention from doing mathematics to conceptualising mathematics, following a development of numerical concepts discussed by Steffe, von Glaserfeld, Richards & Cobb (1983). Pitta, Gray & Christou (2002) discuss the evolution from the use of embodiments (Tall, 2004) to the use of symbolic procepts. Embodiments of the notion of number, the number forms, have been investigated by Galton (1880, 1907) and Seron, Pesenti, Noel, Deloche & Cornet (1992). These are considered embodied objects and according to Gray & Tall (2001) they initially arise in perception. Cobb, Yackel & Wood (1992) argue that internal representations (embodiments) can be a reflection of external representations (physical objects).

§2.3 discusses external representations and their contribution towards the teaching and learning of mathematics. Bruner (1966) believed that representations have a positive impact on learning. Such a stance is opposed by those who argue that their use is more often related with confusion and failure rather than success (Dufour-Janvier, Bednarz & Belanger, 1987; Foster, 2001). Reasons for this can lie in the varied and often unique conceptual structures of individuals who may consequently abstract different things from a "common" representation (see for example, Cobb, Yackel & Wood, 1992; Fuson, Wearne, Hiebert, Murray, Human, Olivier, Carpenter & Fennema, 1997) or the representation itself may not reflect clearly the mathematical idea behind it (Gravemeijer, 1997; Meira, 1998).

§2.4 concentrates on the number line's historical background and its nature. Herbst's (1997) characterisation of the number line as a metaphor of the number system and his definition of it as a consecutive translation of a specific segment U as a unit from zero becomes the focus of our argument. Although the number line's use as an aid to the performance of arithmetic operations is considered (Beishuizen, 1997; Klein, Beishuizen & Treffers, 1998; Anghileri, 2000), its sophisticated nature may lead to complexities related with perceptual (Carr & Katterns 1984; Lesh, Behr & Post, 1987) and conceptual (Davis, Alston & Maher, 1991; Merenluoto, 2003; Hanulla, 2003) problems. These tend to have their roots in the number line's identification as a number track; a distinction made explicit by Skemp (1989).

Lack of such a distinction together with an extensive procedural use seems to dominate within the National Numeracy Strategy (NNS) (DfEE, 1999a), which is discussed in §2.5.2. The implementation of the number line in Dutch school curricula has been a result of reform change, following the Realistic Mathematics Education (RME) theory (Beishuizen, 1999; Treffers & Beishuizen, 1999). In England, the use of the number line within the curriculum, suggests a convenient representation used to model arithmetical processes and understanding its nature seems to be assumed but it is not defined (see for example DES, 1991; DfEE, 1998a; DfEE, 1998b; DfEE, 1999a; DfEE, 1999b).

An analysis of the curriculum guidelines and proposed yearly outcomes within the NNS indicates an extensive procedural rather than conceptual use of the number line. Such use may develop efficiency with a procedure and promote limited conceptual understanding of the number system, but does not appear to encourage the understanding that underscores this sophisticated representation of the number system.

2.2 Theoretical Perspectives on the Development of Mathematical Knowledge

2.2.1 Piaget

The interpretation of children's mathematical activity forms a central component of this study and it is from children's approaches to this activity that qualitative differences in thinking may be inferred. In this sense, the study follows a Piagetian approach. Piaget believed that learning as well as performing mathematics was a matter of active thinking and of operating on the environment and not a matter of passively noting or even memorising what is presented. The basic processes that underpinned the ability to think mathematically he called 'experiences' and he conjectured that these gathered information, "not from the physical properties of particular objects, but from the actual actions (or more precisely their co-ordinations) carried out by the child on the objects" (Piaget, 1973; p. 80). Consequently, for him, activity with objects underscored the development and comprehension of arithmetical relations. He thought that all humans would develop certain structures of thinking as long as they maintained a normal interaction with both the social and physical environment. He used the term 'structure' as a means of describing the organisation of experience by an active learner (Piaget, 1971; pp. 342-345). The system of whole numbers was identified as an example. However, many different mathematical structures could be discovered in this "number" system, for example, the additive group with its rules for associativity and commutativity whilst number itself was seen as part of a larger system that includes fractions. The number line can be seen as a representation of the whole number system and this in turn, part of a larger system that contains fraction - rational, irrational and decimals, whilst its use can emphasise notions of associativity and commutativity.

One of Piaget's concerns was the means through which the co-ordination of actions, which he saw as the roots of mathematical structures, became mental operations and how these operations became structures. An operation, seen as a special kind of mental action, could be reversed. It was an action that can be "internalised" and was destined to become an "interiorized" operation (Beth & Piaget, 1966; p. 206). It could be carried out in thought as well as executed materially; "...actions or operations become thematised objects of thought or assimilation" (Piaget, 1985, p. 49).

Piaget suggested that whilst the co-ordinations of actions and logico-mathematical experiences may interiorise themselves they also give rise to a related form of abstraction, which was termed "reflective abstraction" (see for example Piaget, 1980; pp. 89-97). This form of abstraction reflects both the process through which action is projected to thought or mental representation, and the sense of reorganisation of mental activity, which reconstructs, at a higher level, everything drawn from the co-ordinations of actions (Piaget, 1985; pp. 29-31).

The fundamental assumption of the Piagetian perspective is that new knowledge is in part constructed by the learner through the use of "active methods" and that these "require that every new truth to be learned be rediscovered or at least reconstructed by the student" (Piaget, 1976; p. 15).

2.2.2 Constructivism

Piaget's work established the basis for the constructivist perspective. Vergnaud (1987) described him as the "most systematic theorist of constructivism" (p. 43), while von Glasersfeld (1995) refers to him as being "unquestionably the pioneer of the constructivist approach to cognition in this century" (p. 54).

Jawarski (1988) defined constructivism as

> an abstract philosophical stance about knowledge and its relation to the world and to people's attempts, through their experiences, to try and rationalise the world
>
> (Jawarski, 1988; p. 292).

In his review of constructivism Kilpatrick (1987) argued that the constructivist stance, particularly that of radical constructivism, suggest that "We never come to know a reality outside ourselves" (p. 9). Extending upon this he outlined the two principles of radical constructivism to be:

1. Knowledge is actively constructed by the cognising subject, not passively received from the environment.

2. Coming to know is an adaptive process that organizes one's experiential world; it does not discover an independent, pre-existing world outside the mind of the knower.

In other words, individuals make sense of the world by creating their own knowledge in a way that is meaningful to them. The implication of Piaget's work, and that of the constructivists, is that the knowledge and beliefs that students bring to a given learning situation can influence the meanings they construct in that situation.

From the mathematical perspective, the constructivist view of learning suggests that an individual, by building up and reorganising current mathematical knowledge, actively constructs new knowledge. However, this implies that learning can be a discovery independent from teacher guidance, social interaction and cultural growth (Cobb *et al.*, 1992). Cobb *et al.* argue that in addition to learning's cognitive aspect, a social and cultural nature of mathematical activity exists and that constructivism leads to recommendations that can be the antithesis of learning with understanding. It is not only the way children construct knowledge that is of importance, but also the interpretations they make of instructional devices, the actions with those devices and the communication of mathematical meanings with others.

Though claimed as an over-arching philosophy of teaching, and despite the many useful contributions they have made towards making instruction more interesting and meaningful, it is possible that constructivists have not recognised that there are situations where the adoption of such techniques may fall short (Genovese, 2003). The importance of the Piagetian and the constructivist stance for the perspective of this study rests in the kinds of knowledge that children acquire and how they acquire it.

2.2.3 Types of Mathematical Knowledge

Ryle (1949) suggests that there are two different types of knowledge. He calls the first "knowing how" and relates it to theorising, in terms of mental processes and thinking, associated with "what to do". The second form of knowledge he outlined as "knowing that", which is related to doing and to carrying out physical processes in general. Ryle suggested that the two occur in this order, since "knowing how" can be applied to carrying out processes whilst "knowing that" provides a basis for using the process.

Skemp (1976) placed these two types of knowledge in the context of understanding and he distinguished between two different types: (a) instrumental which is associated with knowing what to do and (b) relational is associated both with knowing what to do and why. Skemp indicated that instrumental understanding had a singular interpretation, which could be identified as the application of "rules without reason". Associated with these two forms of mathematical knowledge, Skemp identified a teaching and learning mismatch. He envisaged situations where a pupil was more in tune with the acquisition of one type of understanding but the teacher placed an emphasis on developing the other. So not only did he identify different forms of understanding but he also suggested that a different emphasis in teaching could result in the teaching of two different kinds of mathematics.

The implication that there may be two types of mathematical understanding leads to the conclusion that there are two types of mathematical knowledge. The identification of such knowledge was a theme developed by Hiebert & Lefevre (1986). They made a distinction between two forms of knowledge; that is procedural and conceptual knowledge. The focus of procedural knowledge is doing through applying and they suggest that procedural knowledge has two components:

(a) the formal language or symbol representation system of mathematics, which involves familiarity with the symbols and

(b) the algorithms or rules for completing the mathematical task, which involves instructions on how to solve mathematical tasks in a hierarchical order.

The essence of the procedural aspects of mathematics within arithmetic is that they focus upon the routine manipulation of objects that are represented in a variety of different ways for example, concrete materials, visual representations, words, symbols or mental images. It is relatively easy to identify whether or not learners are able to carry out procedures with such representations in an adequate way and the resulting performance may be frequently seen as a measure of achievement.

Hiebert & Lefevre suggested that conceptual knowledge makes use of the underlying relationships, which exist within, or between, the objects themselves. They describe conceptual knowledge as knowledge that can be thought of as:

> ...a connected web... a network in which the linking relationships are as prominent as the discrete pieces of information... a unit of conceptual knowledge cannot be an isolated piece of information; by definition it is part of conceptual knowledge only if the holder recognises its relationship to other pieces of information.
>
> (Hiebert & Lefevre, 1986; pp. 3-4)

Conceptual knowledge growth within mathematics may take on two forms. Firstly, it may be the result of the construction of relationships between already existing pieces of information and secondly, it may be the result of the construction of relationships between existing and new information and therefore the new knowledge is connected to existing knowledge.

Sinclair & Sinclair (1986) suggest that the problem with focusing on separate aspects of mathematical knowledge tends to reinforce the belief that the two kinds of knowledge are not only distinct but discrete. Nothing could be further from the truth. We have already seen the strong relationship that Piaget saw between actions and thematised objects of thought. Though actions may form a basic part of arithmetical development, exclusive reliance on them is likely to encourage thinking that is very different to that based on conceptual knowledge. The important issue is the cognitive shift from mathematical processes into manipulable mental objects:

> ... the whole of mathematics may therefore be thought of in terms of the construction of structures, ... mathematical entities move from one level to another; an operation on such 'entities' becomes in its turn an object of the theory, and this process is repeated until we

> reach structures that are alternately structuring or being structured by 'stronger' structures.
> (Piaget, 1972; p. 70)

2.2.4 Developing Sophistication

In the context of the development of understanding in early arithmetic Steffe, von Glasersfeld, Richards & Cobb (1983) conjecture that there is an increasing sophistication in the nature of the counting unit that is used to form the basis for the development of numerical concepts. We may see this unit as an object that supports a counting process and the growing sophistication can be seen in the change in the quality of this object. The change is manifest initially through the substitution of perceptual units, that is countable items and physical objects, and then by the use of figural representations of these perceptual items. Counting can now take place without the presence of the actual items to be counted. Later manifestations of the change may be seen to be motor acts (i.e. pointing or nodding while counting), the use of the number word as a counting unit and finally the use of abstract counting. Here a number word or a numerical symbol can be seen to represent a number of countable items. Now an arithmetical process such as addition or subtraction does not need to start with a specific manifestation of the relevant number, but on the actions and processes that are actually performed upon it. The use of the number line to provide additional experience in counting and in the processes that can be performed upon, it implies that learners have optioned a level of sophistication beyond that of counting in perceptual units.

The changes in sophistication outlined by Steffe *et al.* (1983) provide a sense of the school based experience that can represent changes from a focus on actions to a focus on the "objects" formed from these actions. Here of course the notion of "objects" refers to the entities that form the basis of mathematical action. At the perceptual level, these may be actual things, cakes, pencils, sweets, toys. At a more sophisticated level, they may be the symbols that represent the outcome of the action.

Tall, Davis, Thomas, Gray & Simpson (2000) point out that:

> Once the possibility is conceded that the process-product construction can be conceived as an "object", the floodgates open. By "acting upon" such an object, the action-process-object construction can be used again and again. (Tall, *et al.* 2000; p. 232)

Tall *et al.* make a distinction between "perceived objects", those identified through perception and the resulting descriptions that arise from specific physical manifestations, and "conceived objects", where the focus is no longer on these manifestations but on reflection upon those actions/processes that are performed upon the objects. They suggest that the object-like status of numbers is derived from the total cognitive structure that may be associated with their concept image (Tall & Vinner, 1981). This in turn gives the power to manipulate numerical symbols and to think of their properties.

Such a shift in thinking, which may be seen to be analogous with the shift from actions associated with perceptual units to the use of abstract units (see Steffe *et al.* 1983 above), has been associated with several theories.

Substantial interest in the cognitive development of mathematics has focused on the relationship between actions and entities and has led to theories attempting to account for the transformation of processes into concepts (Davis, 1984; Dubinsky, 1991; Sfard, 1992; Gray & Tall, 1994). These theories have helped to shift attention from doing mathematics to conceptualizing mathematics. Although consensus recognizes that reconstruction as a result of process/object encapsulation or procedural reification does occur, we are still a long way from being able to describe how this is done, although there is evidence to suggest that we may confirm whether or not it has been done. For example, Gray & Tall (1994) suggest that an analysis of a child's interpretation and use of numerical symbolism can provide such evidence in the field of elementary arithmetic. On the one hand, the evidence may point to the substantive use of a procedure such as count-all, whilst on the other it may be exemplified by the use of known facts to establish unknown ones.

Though these theories have intrinsic differences, they share common ground in that they attempt to account for the cognitive reconstruction that underscores the development of conceptual thinking in mathematics. However, it is suggested that the individual's perception and interpretation of the original objects and the actions that may be associated with them will influence the quality of this development. Objects have different facets and therefore are subject to different interpretations. For example, counting starts with objects perceived in the external world which have properties of their own; they may be round or square, red or green or both round and red. However, these properties need to be ignored (or at least temporarily disregarded) if the counting process is to be encapsulated into a new entity — a number that is named and given a symbol — that may then be associated with new classifications and new relationships.

Sfard (1991) suggests that there are three stages in the development of the ability to use newly constructed objects as a basis for a new series of procedural transformations. She suggests that two phases are associated with operational thinking (which may be seen to be closely associated with procedural thinking). The first is the process of interiorization, which is a phase during which the learner becomes familiar with a process. The second phase involves the notion of condensation — the process is compressed into an easily manipulable entity. It is during this phase that there occurs the possibility to raise the quality of thinking through the notion of reification — operational thinking is raised to a new level of thinking through the cognitive reconstruction of the process and outcome of operational thought. Reification involves a cognitive shift that sees the creation of a new entity that can be thought about as a whole but detached from the processes that gave rise to it.

Similarly concerned with the transformation from process to object, Dubinsky (1991) identifies the notion of encapsulation "or conversion of a (dynamic) process into a (static) object" (p. 101). He suggested that Piaget was applying the idea when he referred to "... building new forms that bear on previous forms and include them as contents" (p. 101).

The theories of Skemp, Hiebert & Lefevre and Sfard can engender the perception that different notions of mathematical understanding are dichotomous. However, though it is possible to see, for example, learners clearly demonstrating procedural understanding in the absence of any conceptual understanding (see for example Gray 1991) it is less easy to identify students with conceptual understanding who do not possess procedural understanding. The relationship between the formation of concepts from interiorised actions explored by Sfard and Dubinsky indicates that reified or encapsulated objects can be back referenced to the actions that were at the foundation of their formation although Sfard implies that this is not necessary since the object can become completely detached from the process that formed it. Skemp's notion of relational understanding implicitly contains reference to the possession of instrumental understanding – "Knowing what to do and why"– whilst Hiebert & Lefevre's notion of conceptual understanding explicitly contains reference to the inclusion of procedural understanding. Thus, far from being dichotomous, the more sophisticated notion within each theory implicitly contains the less sophisticated. Indeed, this is a theme discussed by Sfard (1991) who rejected the ostensive incompatibility of processes and objects to emphasise their compatibility.

It was partly in response to her question "How can anything be process and object at the same time?" (Sfard, 1989; p. 151) that Gray & Tall (1994) suggested that such ambiguity was a feature of mathematical symbolism. They suggest that evidence that different interpretations could be placed on symbolism had been mounting for some time (see for example Tall & Thomas, 1991; Gray, 1991). Their observations had indicated that learners displayed qualitative differences in their interpretation of mathematical symbolism. Some emphasised the extensive use of procedural approaches through which they had to consciously cope with the duality of process and concept whilst others appeared to use the flexibility that could be associated with the duality to make an appropriate choice. At the core of these differences, they suggested, was "the amalgam of concept and process represented by the same symbol" (Gray & Tall, 1994; p. 6). The consequence was that they furnished "the cognitive combination of process and concept with its own terminology", the notion of procept, an amalgam of three elements "a process which produces a mathematical object, and a symbol which is used to represent either process or object" (Gray & Tall, 1994; p. 6).

With particular reference to arithmetical procepts, for example the symbol 3 can be counted, spoken, heard, written and read, and grow in richness of meaning so that it not only represents procedural aspects such as counting, but also conceptual relationships where the object 3 can be represented by different symbols. Gray & Tall (2001) later indicated that arithmetical symbols:

> act as pivots between processes and concepts in the notion of procepts and provide a conscious link between the conscious focus on imagery (including symbols) for thinking and unconscious interiorised operations for carrying out mathematical processes
>
> (Gray & Tall, 2001; p. 67)

In making this claim they do so to draw the distinction between the notions of symbolic procept and embodied object which begins with the mental conception of a physical object in the world as perceived by the senses "...and can only be constructed mentally by building on the human acts of perception and reflection" (Gray & Tall, 2001; p. 67).

2.2.5 Embodied Objects

Ideas about embodiment of thought have their roots in theory that reinterprets the basic organisation of living things and the way in which they interact with their environment. In a sense therefore, it is not unsurprising that many of the current ideas can be traced back to biological foundation for the explanation of cognitive phenomena (Maturana & Varela, 1998). Maturana & Varela's theory concentrates on a human perception of the world that attempts to capture objective reality through studying cognition as embodied action. The use of the word action emphasises that perception and action are not only fundamentally inseparable in lived cognition, but also evolve together.

Lakoff & Johnson (1980, 1999) attempted to establish a general description of human understanding concentrating on embodiment in the context of conceptual structures. They widen the scope of the relationships inherent in Maturana & Varela's notion to include language, meaning and conceptual thinking and an important feature to emerge from their theory is that of image schema identified as embodied concepts that result from the mental-images learned through our bodily interaction with the world.

The contribution that ideas associated with embodied cognition can make to mathematics education has featured regularly in the literature over the past decade (see for example, Sfard, 1994; Núñez, Edwards & Matos, 1999; Gray & Tall, 2001; Tall, 2002; Watson, Spirou & Tall, 2003).

Núñez, Edwards and Matos (1999) indicate the way in which they see the relationship between notions of embodiment and mathematics claiming that

> ...embodiment provides a deep understanding of what human ideas are, and how they are organized in vast (mostly unconscious) conceptual systems grounded in physical, lived reality. (Núñez, Edwards & Matos, 1999; p. 50)

Wilson (2002) suggests that:

> ... while a cognitive process is being carried out, perceptual information continues to come in that effects processing and motor activity is executed that affects the environment in task relevant ways (Wilson, 2002; p. 2)

In the context of this study, this presents us with an important caveat. We may see embodiments as pre-loaded representations acquired through prior active, linguistic and relational experience with the environment.

Within the field of elementary arithmetic the theorised encapsulation or reification of a process as a mental object is often linked to a corresponding embodied configuration of the objects acted upon. These Gray & Tall (2001) identified as base objects. Counting processes initially operate on physical objects. Thus, the seemingly abstract concept of number already has a primitive existence in the physical configurations of base objects.

Translating this into the context of Steffe *et al.* (1983), we may see that a child's development within elementary arithmetic will initially make use of base objects that are actual physical objects. Later these may become figural in the sense that they are representations of these physical objects. Later still, these are subsumed within an encapsulated counting process, which is compressed into the concept of sum.

This study's effort to consider children's understanding of the use of the number line will draw upon a distinction that is to be made in terms of the use of external or internal representations. The latter suggests the notion of mental representation and in the sense that is discussed within the study these mental representations may be embodied perceptions of the number line. Gray & Tall provide a rational for consideration — a mental conception of a finger is an embodied object, but the actual physical object 'finger' is not. A mental image of 'five fingers' clearly is an embodiment, but the imagery for the number 'five' is not. On the one hand, we may see the use of external resources to assist the mental representation and manipulation of things that are not present. On the other, we may see the purely internal uses of sensori-motor representations in the form of mental simulation. It is within this sense that Tall (2004) describes his notion of embodiment:

> ... my own notion of 'embodiment' relates to how we consciously embody concepts in visuo-spatial ways, corresponding to ways in which we embody an abstract concept by giving it a familiar concrete referent. (Tall, 2004; p. 7)

Such a view leads us to consider embodiments of number that arise from the role of imagery.

2.2.6 Embodiments of the Notion of Number

Mental representation, particularly in the form of images has a long history starting with Aristotle whilst Pavio (1986) has suggested that the latter is the most persistent of all representational concepts. The work of Galton (1880a, 1880b, 1880c), though subsequently criticised because of its methodology (Philips, 1897; Spalding & Zangwill, 1950) drew attention to mental representations associated with the notion of number. He described these as "number forms" and the greater proportion of his reported number forms may be identified as a version of the number line though most were not associated with a single straight line. Though Galton (1907) recognised that some of his respondents may have exaggerated their forms, many of his claims about mental imagery associated with numerals, are not controversial though they varied between subjects, some parts of the number form are clearer than others and small numbers are more spaced out than large numbers.

In more recent research, Dehaene (1997) also suggests logarithmic patterns of adults' internal representations of numbers, where larger numbers tend to be compressed into a smaller space. Siegler and his colleagues (Siegler & Opfer, 2003; Siegler & Booth, 2004; Siegler 2005) studied children's estimations of numbers in the range 0 to 100 and suggest logarithmic patterns for younger children and linear patterns, where all numbers are equidistant, for older children.

In a partial replication of Galton's work, who concluded that 1 in 30 adults and 1 in 21 youths have number forms, Seron, Pesenti, Noel, Deloche & Cornet (1992) identified that approximately 9% of the population as a whole reported mental images associated with the use of the number line. The qualities of these lines often illustrated a left to right progression of the succession of numbers with either a positive inclination or occupying different planes. Some number lines would have equally spaced marks for every number, others would have only some numbers marked with many numbers missing. These could go up to 100 or more. Seron *et al.* concluded that although number forms seem to exist and there is evidence that they might be used in number and calculation processing, their function is not clear.

Galton had concluded from an analysis of the responses of his subjects (who ranged from public school boys to eminent scientists) that the number forms had been present for as long as his subjects could remember. Seron *et al.* confirmed Galton's results by suggesting that visual patterns or number forms emerged during childhood and that they remain the same in time. In contrast, while Ernest (1983) believes that number lines are not self-created but learned, Dehaene (1997) argued that number forms may be a conscious and enriched version of an innate number line that everybody possesses.

Thomas, Mulligan & Goldin (1996) attempted to gain an insight into children's conceptual understanding of number by studying children's structural features of internal representations via their expressed external imagery. They suggest that imagery can contribute to a better numerical understanding and to the formation of a more flexible way of thinking about numerical concepts. Among their concluding remarks, they propose that low and high achievers have qualitatively different images. The former appear to have weak visualisation skills and produce representations lacking in mathematical structure, constituting of poorly organised objects as units. These would be pictorial and iconic images, based on visual imagery and describing a sequence of visual representations such as animals or quantities of objects. High achievers have structural as well as dynamic (changing or moving as opposed to static which is fixed) imagery and produce representations of mathematical structure. Mulligan, (2004) suggested that the representation of mathematical structure is of high importance in mathematical understanding. She defined mathematical structure as the structural development of the number system, for example knowledge of the fact that the units on a ruler or a clock are equal in size, which indicates conceptual understanding.

Galton's (1907) number forms, Seron *et al.*'s (1992) mental representations of numerical objects as well as Thomas, Mulligan & Goldin's (1996) reported imagery for numeration may be seen as embodied objects. Their subjects reported seeing simple digits or numbers, numbers transformed into patterns as found on a dice, numbers with colour and numbers as on a number line. Seron *et al.* suggest that quantity directly represented by "patterns of dots, or other things such as the alignment of apples or a bar of chocolate" (p. 168) may be deemed to be analogical. In Gray & Tall's (2001) sense these objects are embodied — they arise initially in perception but they can carry mental ideas (i.e. the dots on the dice may carry the idea of five). Clearly, a mental conception of a number line is an embodied object and of course, an actual physical number line is not.

2.2.7 Summary of Section 2.2

Piaget's theory of constructing knowledge by operating on the environment can be seen as an initial stage towards the evolution of the embodiment theory. Within the field of elementary arithmetic the theorised encapsulation or reification of a process as a mental object is often linked to a corresponding embodied configuration of the objects acted upon (Dubinsky, 1991; Sfard, 1992; Gray & Tall, 1994). It is clear that there is a growing sophistication in the nature of these entities, from physical objects to mental operations with the number symbols themselves. This development is manifest in an increasing detachment from immediate experience. Within elementary arithmetic this increasing detachment can be seen in the evolution of different aspects of counting and a change in the form of unit counted (Steffe, Richards, von Glasersfeld & Cobb, 1983) and consequently in a change from the use of perceptual objects through embodiments in the form of words and mental representations of physical objects to the use of symbolic procepts (Pitta, Gray, & Christou, 2002). Cobb *et al.* (1992) suggest that internal representations can be a reflection of external representations in the mind, arguing that there has to be an already existing conceptual substance in the child's mind, for the child to relate an external representation to something.

> (For learning to take place) ... students modify their internal mental representations to construct mathematical relationships or structures that mirror those embodied in external instructional representations. (Cobb *et al.*, 1992; p. 2)

2.3 External Representations

2.3.1 A View from Bruner

During the decade of the 1960's a period of curriculum re-evaluation and reform took place internationally. Its main concern was the teaching of the basic mathematical concepts to children taking into consideration the child's cognitive capabilities. Bruner studied children's cognitive processes and their mental representations of the concepts they were learning since he believed that the use of external representations in mathematics contributes significantly towards meaningful learning. (Resnick & Ford, 1981)

In order to achieve meaningful learning, Bruner (1966) suggested a specific path of knowledge, consisting of three stages of increasing sophistication (modes of representation), through which the individual's concept knowledge is conserved: (a) the enactive stage (conservation of actions), (b) the iconic phase (conservation associated with pictorial imagery) and (c) conservation through symbolic means (for example a word or a mark that stands for something else). However, even individuals with a fairly sophisticated understanding of mathematics may utilise each mode of conservation to different degrees. Whereas an individual who conserves knowledge only through enactive representations has only one choice of interpretation available, an individual who can see the relationship between the three has flexibility and choice in those situations that require the knowledge.

2.3.2 Pedagogic Representations

Dienes (1963) argues that children may learn mathematics from their own experience by going through a transition from "play" to cognitive processing. Based on this view, the Dienes' base ten apparatus (Dienes blocks) was designed to represent the concept of number through place value and its purpose was to help children understand how the number system works. Dienes raised the issue of children's failure to abstract the desired concept and constrain themselves to the material being handled at the moment, hence not being able to generalise to additional materials. In order to overcome this, he suggested presenting children with a variety of materials that are as different as possible, so that children view the structure from different perspectives and eventually abstract the common concept of interest (Dienes, 1963).

Representations of mathematical ideas may have different qualities and these differences were categorised by Lesh, Post & Behr (1987) (p. 33):

1. experience-based "scripts" – in which knowledge is organized around "real world" events that serve as general contexts for interpreting and solving other kinds of problem situations

2. manipulatable models – like Cuisenaire rods, arithmetic blocks, fraction bars, number lines, etc., in which the "elements" in the system have little meaning per se, but the "built in" relationships and operations fit many everyday situations

3. pictures or diagrams – static figural models that, like manipulatable models, can be internalized as "images"

4. spoken languages – including specialized sub-languages related to domains like logic, etc.

5. written symbols – which, like spoken languages, can involve specialized sentences and phrases.

They suggested that representations of mathematical concepts or processes become the mediators of mathematical ideas between the teacher and the student, arguing that external representations enable a two-way communication: (a) from the teacher to the pupil in terms of the mathematical ideas and (b) from the pupil to the teacher in terms of the child's understanding. It is interesting that Lesh *et al.* suggest that the number line possesses elements that have little meaning and yet built in relationships and operations that can be used. It is an assumption of this study that the elements of the number line have a deep sophisticated meaning (§2.4.2) and that relationships that are used are dependant upon this meaning (§2.4.3).

The way in which apparatus is used in classroom as a means of communication was considered by Foster (2001) who concluded that teachers used it in two distinct ways. The first of these he identified as the demonstration mode — the teacher uses the apparatus in class for demonstration both as a precursor to and as a foundation of written methods (pencil-and-paper). Such a mode can be used for communication between children and teachers but it may also rely extensively on memory and lead to instrumental understanding. Foster suggested that the second mode — developmental mode — gave children the opportunity to use the apparatus freely and creatively according to personal idea and understanding so that they could make sense of the apparatus. This mode is implicit in Dienes' recommendations as to the use of his blocks.

Using a representation as metaphor to support thinking — a "helping tool" suggests that there is the need for learners to know and understand the nature of such a tool in order to able to use it flexibly and with understanding. Cobb *et al.* (1992) identify a "learning paradox" between the pedagogue's intention in the introduction of a "helping tool" and the learner's interpretation of it by drawing attention to the difficulties that the uninitiated have in picking out, from the multitude of alternatives, the relationships that are self evident to the initiated. The picture emerging from their critique is that student interpretations of instructional materials are not always compatible with what the teacher thinks is being shared with the students.

This is a theme that was earlier considered by, Dufour-Janvier, Bednarz & Belanger (1987). They stress that there can be a difference between the motives of the learner and the intentions of the person suggesting the use of a representation. They suggest that promoters of particular types of representation have great expectations for the outcomes of their use in the learning of mathematics. Advocates of representations expect that the learner will:

(a) perceive the representations as mathematical tools,

(b) be able to choose the appropriate representation and justify why,

(c) know and abstract the properties and capabilities of every representation and construct the common concept,

(d) apply the knowledge of a concept in different contexts.

However, they suggest that although children may have studied or used a representation in the past, this does not necessarily mean that they will choose the same representation again when it comes to solving problems. Children tend to perceive representations as isolated materials and do not link them with solutions of problems nor do they see a common embodiment associated with different representations.

> We note in fact that in current teaching the use of representations frequently does not lead to the attainment of the desired objectives, and that in certain cases their contribution is nil. Serious doubts need to be raised on the use that is made of representations especially when, as we have observed, certain representations lead more to difficulties rather than functioning as aids to learning. (Dufour-Janvier, Bednarz & Belanger, 1987; p. 116)

Reasons for such outcomes were suggested by Hall (1991), who argued that although mathematics educators may insist on the use of concrete representations for the teaching and learning of mathematics, they may not clarify the relationship between the two. Though we may see such claims such as:

> Concrete apparatus like beads and cubes, balances and rulers, will still provide sound opportunities to handle and view the models that are the basis of mathematical relationships. (Anghileri, 1995; p. 7)

> ... the number line appears to be an ideal representation to help students connect whole-number and fraction processes such as counting. (Baturo & Cooper, 1999; p. 82)

the way such claims are applied by teachers within classroom may not give the intended outcomes. The relationship between the advocate of the representation that is the number line, in this case the National Numeracy Strategy, and teaching and learning associated with its use is the underlying theme of this study.

However, what seems to be more frequently identified within literature is the problematic use of instructional materials (Carr & Katterns, 1984; Dufour-Janvier, Bednarz & Belanger, 1987; Davis, Alston & Maher, 1991; Cannon, 1992). Hall suggests that the pedagogic value of such materials relies on children's understanding of mathematical concepts through their use and on the way teachers use them. Meira (1998) on the other hand, suggests that the effectiveness of an instructional material depends on how well the physical material itself reflects the mathematical meaning or idea behind it and on the quality of what is abstracted by the child from the use of the material.

Children's mental constructions vary and therefore children abstract different things from the use of a "common" external representation and consequently they can construct new knowledge in different ways (Cobb *et al.*, 1992; Bills, 2001; Tall, 2004). Kilpatrick (1987) suggests that the meaning a representation has for an individual is purely subjective. Fuson, Wearne, Hiebert, Murray, Human, Olivier, Carpenter & Fennema (1997) conclude that children's conceptual structures vary and what a child perceives or abstracts from a "common" classroom experience, depends heavily on his/hers unique conceptual structures.

Though he is investigating children's use of Dienes 'apparatus', it is conjectured that Foster's (2001) conclusion in that context may be equally applied to the use of the number line and is therefore worth considering here. Foster identified four types of learning associated with the use of Dienes apparatus: (a) Mental Symbol Manipulators whose sophistication in arithmetic is such that they already perform it mentally and impose their knowledge structure on the apparatus, (b) Mental Apparatus Manipulators, who have less sophisticated mental structure yet work with the apparatus in such a way that they abstract the structure and are able to use it mentally in other contexts, (c) Physical Apparatus Manipulators who are able to perform the arithmetic tasks with the apparatus, but this does not transfer to other contexts and (d) Basic Apparatus Manipulators who are not able to perform the arithmetic tasks with the apparatus. Thus, he argues that different children may use apparatus in different ways, using it appropriately or inappropriately or not at all. He concludes that instructional displays with the apparatus are meaningful only with respect to the learners' activities and they might not be suitable for all children in the same way.

> The real situation is that there are ways in which apparatus is used which might prove helpful to children at different points of development and in different ways... It does need to be noted that at any one point apparatus can offer specific help or erect significant obstacles to understanding. (Foster, 2001; p. 208)

He suggests that the use of apparatus may "get in the way" of the learning of mathematics. Though children may learn how to use a particular representation, the mathematics the representation represents may not be developed (Dufour-Janvier, Bednarz & Belanger, 1987; Kilpatrick, 1987). Gravemeijer (1997) attributes the failure to the fact that "...the mathematics embedded in the models is not concrete for the students." (p. 315). An issue for this study will be how is the mathematics embedded in the number line understood by the children.

2.3.3 Summary of Section 2.3

The picture that has emerged is one that suggests that although pedagogic representations help in the teaching of mathematics, they can also be problematic in learning. Viewing learning from a constructivist's perspective, we can see that every child has their individual mental constructions on which they may base their abstractions and consequently their creation (or otherwise) of new mathematical concepts. Children may abstract different meanings from what seems to be a "common" representation within a class and later construct concepts that are only meaningful to the individual (Dufour-Janvier *et al.*, 1987; Fuson *et al.*, 1997). Trying to homogenize learners by using a representation that seems appropriate and is meaningful to an adult might not always be the cure for the learning of a particular mathematical concept (Cobb *et al.*, 1992). Since the focus of this thesis is the representation that is the number line, we now turn to see what it represents and how its use is advocated.

2.4 A Specific Representation: The Number Line

2.4.1 Historical Background

The number line does not receive an explicit reference within texts on the history of mathematics (Smith, 1923; Smith, 1925; Heath, 1981). Its closest reference is the ruler, with Edmunt Gunter's invention of the "line of numbers" in 1620 and William Oughtred's circular slide rule in 1621 (Gullberg, 1997). However, an 11th century Peruvian recording instrument, the Inca *quipu*, is related to the number line in that it combined the idea of a straight line (represented by a length of string) and points on the line (knots on the string at different places) to represent units or decimals and the absence of knots to represent zero (Karpinski, 1965; Menninger, 1969; Gullberg, 1997).

Nevertheless, the association between number (real number) and line has been evident since Babylonian times (Wilder, 1968). The Greeks intuitively conceived real numbers as corresponding to linear magnitudes. The Greek idea of "magnitude", which is substituting magnitude for number, implied that one may think of "numbers as measured off on a line" (Bourbaki, 1984, p. 121). The number line is, therefore, an abstraction of a representation strongly associated with the notion of a measure instrument since continuity underscores it. Starting from the Euclidean line, a "sense of continuity" can be created for and by the individual and the result be used as a number line to represent natural numbers.

2.4.2 What is the Number Line?

In the context of school mathematics and somewhat in contrast to the above statement, the Association of Teachers of Mathematics (ATM, 1967) referred to the number line as if it is a tactile object, and suggested "cutting" it and "wrapping" or "stitching" it round the circumference of a circle. The number line was presented as a result of mappings with particular attention given to the visual effect created by the varied-sized jumps on it.

In 1979 the Department of Education and Science (DES) in England, published a Handbook of Suggestions in Mathematics for the ages 5 to 11 and stresses that the number line is extremely important during the early stages of mathematics for addition and subtraction. Structured number lines and empty number lines are presented within the suggestions as equidistant points for natural numbers. These suggestions contradict later conceptions of the empty number line's property, which maintained number order but not the scale (Dettmer-Kratzin, 1997).

Williams & Shuard (1970) extensively discussed the nature and use of the number line in primary mathematics classrooms as a representation of the number system. They point out that it can support conceptual understanding in number development and specify that its underlying continuity highlights the qualities that indicate that each point on the line corresponds to a unique number. Between the natural numbers, there are an infinite number of points that correspond to other numbers such as fractions and decimals. They emphasise the importance of orderly and equally spaced points and the possibility of extending the number line in both directions to infinity.

Gullberg (1997) defines the number line as a graduated length in unit distances, extending to negative infinity and positive infinity on either sides of zero. All real numbers can be mapped on this line. At a more sophisticated level, Herbst (1997) concurs that the number line is a metaphor of the number system and he defines it as the consecutive translation of a specified segment U as a unit, from zero. U itself can be partitioned in an infinite number of ways (i.e. fractions of U) and to form a number line:

> one marks a point 0 and chooses a segment u as a unit. The segment is translated consecutively from 0. To each point of division one matches sequentially a natural number. (Herbst, 1997; p. 36)

It is from such a definition that we may see why Herbst confirms that the number line can be used to represent the number system. All kinds of numbers can be represented on it. For example, natural numbers (1, 2, 3, ...), integers (..., -2, -1, 0, 1, 2,..), rational numbers (p/q, where p and q are integers),and real numbers (having infinite decimals). It is this quality that enables Herbst to write about what he calls the "number line metaphor" and the "intuitive completeness" (Herbst, 1997, p. 40) of the number line, evolving from plane geometry. It is therefore possible to build a series of different number lines to introduce different numbers of the number system. Start off with the natural numbers number line, carry on with the positive rationals number line, then the integer number line, followed by the negative rationals number line and finally introducing the real numbers number line, which would include all numbers. In this way, the number line could be in one-to-one correspondence between numerical statements and number-line figures. It is these features that would appear to suggest the use of the number line as a pedagogical tool.

Of course, in its completeness, the number line is a very sophisticated metaphor and recognising this, sophistication can grow through experience and use. The sophistication that would arise from understanding Herbst's definition could be available at a simplistic level for the child who is dealing only with whole numbers and at a more complex level, for the adult who has experienced the range of numbers that lead to a sense of the "real" number line.

Behr & Post (1992) define the unit of measure on a number line to be the distance from 0 to 1. In order to create multiples of the unit, iteration of this specific 0 to 1 distance is needed, while to create subunits, the distance 0 to 1 would have to be partitioned into equal lengths. They stress that children should be able to represent the same fraction at different places on the number line. Bright, Behr, Post & Wachsmuth (1988) outline the creation of a number line as a sequence of activity; identification of a length representing the unit, iteration of the unit, subdivision of the unit. They indicate that this development can reinforce the notion of continuity, but that to make sense of any point on the line, two other reference points need to be included. They confirm that the number line combines visual and symbolic information.

2.4.3 Use of the Number Line in Classroom

Though it is clear that the number line possesses certain underlying features, there have been a range of suggested modifications and uses of it as a representation to support the development of mathematical skill and understanding.

Freudenthal (1973), for example, recommended that vertical number lines or even inclined number lines should be used with young children, since the traditional number lines we are used to are indeed vertical (i.e. thermometer, the instrument to measure body length) and it is for this reason, he argues, the horizontal number line will naturally not work properly with children. Anghileri (2000) recognises that there may be various versions of the number line (number track, calibrated number line, empty number line) and recommends the use of a horizontal straight line.

Harries & Spooner (2000) argue that there are two types of number lines: the numbered number line and the empty number line. Both types are horizontal lines, with the distinction that the former has marks indicating the intervals between numbers, while the latter has no numbers, marks, beginning or end points. The authors refer to the numbered number line as a "static reference tool", where everything is visible for the pupil, while the empty number line (ENL) is a "thinking tool" which connects mental with written work.

Successful results from an experimental empty number line project (Klein, Beishuizen & Treffers, 1998) in the Netherlands suggest that the use of the empty number line for the learning of addition and subtraction in primary schools is a very powerful model and its power is attributed to psychological and didactical reasons — it can support the clear representation of problems although it is not "structured" in the sense that it has no marks for every single number. In this way, it avoids the passive finding of an answer, while it encourages the development of new more sophisticated strategies and particularly those that may be associated with partitioning, associativity and commutatitivity.

Rousham (1997) and Bramald (2000) considered that the number line can act as a diagnostic tool for both the pupil and the teacher. It can be a helping tool for teaching whole number operations (Ernest 1983, 1985; Carr & Katterns 1984; Beishuizen, 1997). The fact that pupils can physically act on it allows them to be cognitively involved in their own actions and make decisions about where to make a mark and write down the numbers on the line (Beishuizen, 1997; Klein *et al.*, 1998). Behr, Harel, Post & Lesh (1992) link the use of the number line to the following activities:

> Associate whole numbers, fractions, and mixed numbers with points on the number line. Convert improper fractions to whole or mixed numbers. Determine equivalence. Add fractions with same denominators. (Behr, Harel, Post & Lesh, 1992; p. 329)

Discussion of the different strategies used by different individuals within a classroom can be encouraged by its use and this may therefore activate the children's mental thinking (Treffers & Beishuizen, 1999). Thompson (1999) suggested that the ENL could take the form of a "vehicle" to support mental calculation strategies, a feature supported by Harries & Spooner (2000) and in evidence, though not made explicit, within the National Numeracy Strategy (NNS) (1999a).

However, there are dangers associated with the ENL. Pupils may hang on to it for too long and therefore have a lot of support but very little mental activation (Beishuizen, 1985, 2001). Cannon (1992) suggested that the number line is not particularly easy to interpret and use by children within Grades 5 and 7 (10 and 12 year-olds). He further suggests that children's misconceptions associated with the number line could be attributed to perceptual and conceptual factors.

2.4.4 Perceptual Problems

Children's difficulties with the conception of the number line may arise from two main sources; the representation that is provided for them to consider and the interpretations and actions that they associate with it as both a conceptual and a procedural model.

Freudenthal (1973) provided a good summary of problems with the former:

> I must explicitly draw attention to another mistake I have found in books and films where the number line was represented. The number line is pictured as many rulers, with partition points or strokes, and at their left the numbers 1, 2, ..., of course without a 0. This calls forth all the well-known mistakes children can make when using the ruler (measuring from 1 instead of 0). I even saw a number line, where the numbers 1, 2, ... were exactly midway between the partition points. (p. 256)

Anghileri (2000) seems to miss the point of such a comment with her suggestion that the number track may be identified as version of the number line:

> Initially the number line may be represented by linking beads on a string, or with cubes that can be structured into a line that is sometimes referred to as a number track... This number line appears on rulers and scales, ... (p. 10)

Such a perception may cause considerable confusion for those who are not aware of the conceptual differences between the two.

Dufour-Janvier, Bednarz & Belanger (1987) argue that "the premature introduction of representations can sometimes explain their uselessness, and furthermore they may even have negative effects on learning" (p. 116). Among the examples they provide there is one concerning the use of the number line for the learning of positive integers as stepping-stones with a hole in between two successive stones. Their results suggest that this may have an impact on students' perception of the density concept of real numbers, since with such use the student may imply that there are no numbers between two whole numbers.

The number track does not possess the underlying feature of continuity that is the governing feature of the number line. Its replicated units represent the discrete aspect of number and not its continuous aspect. Skemp (1989) distinguishes between the terms number line and number track:

> The differences between a number track and a number line are appreciable, and not immediately obvious. The number track is physical, though we may represent it by a diagram. The number line is conceptual – *it is a mental object, though we often use diagrams to help us think about it.* The number track is finite, whereas the number line is infinite. However far we extend a physical track it has to end somewhere. But in our

> thoughts, we can think of a number line as going on and on to infinity.
>
> (Skemp, 1989; pp. 139-141; Italics not in the original text)

The notion that a number line can be empty can also be the source of considerable confusion. It is not a number line until reference points are identified. Dufour-Janvier *et al.* (1987) suggest that placing the numbers at equal distances on the number line is essential for the measure concept, whilst Streefland & Heuvel-Panhuizen (1992) hypothesize that children's difficulties with the conception of the number line arise because of their incomplete sense of measurement. It may be hypothesized that competence with what is termed the empty number line may provide very little indication of an appreciation of the relationship between the number line and its role as a representation of the number system.

Bright, Behr, Post & Wachsmuth (1988) and Behr & Post (1992) suggest that the number line is currently an extensively used model in the teaching of mathematics in elementary school, and whilst generally effective is also the source of difficulty both in instruction and its use by children. They indicate that the number line can help children learn about addition and subtraction and that it can strengthen children's understanding of fraction order and equivalence, as long as it is not the first model used. This latter point is supported by Dufour-Janvier *et al.* (1987) who stress the importance of the use of a representation at the right time:

> The premature use of a representation, as well its application in an inappropriate context, can lead children to develop misconceptions that will hinder them in later learning.
>
> (Dufour-Janvier *et al.*, 1987; p. 117)

A common mistake is to count the marks or the symbols/digits on a number line and ignore the importance of the spaces (Carr & Katterns, 1984; Baturo & Cooper, 1999; Bragg & Outhred, 2004). In his study, Cannon (1992) concluded that points on the lines given to the children were perceptually salient and a conception of counting was not fully constructed on the pupils' behalf. Lesh, Behr, & Post (1987) suggest that perceptual distractors influence students' rational number thinking and they provide an example where students had difficulty to find a third on an already partitioned bar compared to a non-partitioned bar. Such confusion may be attributed to the lack of distinction between number line and number track, such as the following:

> On the number track, numbers are initially represented by the number of spaces filled, with one unit object to a space. So it is rather like a set loop, in which the number of objects is automatically counted. *Even if physical objects are not used, it is the number of spaces counted which corresponds to a given number.* So the number zero is represented by an empty track, corresponding to the empty set. *The number one is represented by a single space filled, which means that the first space on the number track is marked 1 and not 0.*
>
> (Skemp, 1989; pp. 139-141; Italics not in the original text)

Carr & Katterns (1984) argue that understanding and application of the number line principles may be neglected because teachers over-emphasise mechanical procedures on the line and they suggest that a way to avoid this is to guide the pupil's direction towards the size of the "jumps" and not the digits under the end of each "jump".

2.4.5 Conceptual Problems

It would appear that the number line is not particularly beneficial when it comes to the teaching and learning of fractions — broadening the scope of the number line from whole numbers to the rational numbers may present difficulties. Davis, Alston & Maher (1991) have argued that the number line may be compatible with the pupil's already existing knowledge of, for example, whole numbers but it may be inconsistent with new knowledge (i.e. mapping fractions on the number line). Hartnett & Gelman, (1998) suggest that what children have learned in ordering natural numbers on the number line does not apply when they need to order fractions. When ordering natural numbers, numbers get bigger as the values increase. The opposite happens with fractions, when the bigger the denominator the smaller the fraction. Children's existing knowledge no longer applies and this may automatically obstruct an expansion of the child's development of the number system. Children may be able to use methods in particular cases but do not always transfer their knowledge and apply it on another concept, therefore resulting in error and failure (Behr, Lesh, Post & Silver, 1983; Bright, Behr, Post & Wachsmuth, 1988; Foster, 1993).

A cognitive conflict can be created for the learner when from the discrete nature of natural numbers (reinforced with the use of the number track) (s)he progresses towards the compact nature of rational numbers and the continuum of real numbers (Merenluoto, 2003). Merenluoto indicates that Grade 7 Finish students had difficulties in perceiving and locating correctly (even approximately) a fraction as a number on a number line and they were unable to identify the whole correctly. The minority of the students who solved the given task (pinpointing 3/4 on a segment 0 to 1) correctly used the strategy of halving the 0 to 1 segment in the middle and then the segment 1/2 to 1 again in the middle. Hanulla (2003) indicated that very few students transformed fraction into decimals on the number line and make the link that though they identified equivalent points they were the same numbers. Merenluoto & Lehtinen (2002) indicated that high achievers have a higher level of understanding of the density of the number line in comparison to low achievers. The latter tend to think more operationally when it comes to the density of the number line and they make no reference to the structural differences between the numbers.

Children's conceptions of the use of the number line to initially develop then portray thinking in the context of whole number operations such as addition and subtraction, and their conceptions of the number line and its representation of fractions and decimals in relation to teaching will be considered within this study.

2.4.6 Summary of Section 2.4

The number line is a sophisticated model that, through its dense nature and inherent continuity, theoretically can be used as an appropriate representation to fit students' development of the number system (Herbst, 1997). It can be a model that provides a perspective on the development of particular number concepts and on the relationship between these concepts. It can also be used as a tool to support the development of arithmetical operations (Beishuizen, 1997; Klein *et al.*, 1998). Nevertheless, the concerns and issues associated with the use of pedagogic representations (see §2.3.2) apply to the use of the number line. Perceptual and conceptual factors may lead to problems in its use, particularly when it involves the reconstruction and consequent extension of the concept of number from whole number to fraction and/or decimal (Lesh, Behr & Post, 1987; Davis *et al.*, 1991; Merenluoto, 2003).

In the Netherlands the number line and a modified number line, the empty number line, has been used as a result of reform change and it appears to have led to success in children's performance in arithmetical operations and the development of mental strategies. The outcome from this initiative may have had some influence on the development of the National Numeracy Strategy and it is these two initiatives that are the subject of the following section.

2.5 Concerns with Achievement

In the Netherlands during the 1970's, the issue of changing the teaching of mathematics was raised because of a need to consider an improvement in standards and the quality of mathematical understanding. Hanz Freudenthal introduced the term "educational development", which not only implied curriculum development, but also aimed to modify educational practice (Gravemeijer, 1994). Van Den Heuvel-Panhuizen (2002) indicated why the nineties could be seen as the decade of standards in the Netherlands and describes the situation regarding mathematical achievement and the need for a curriculum initiative. She indicated that a crucial aim of the new learning-teaching methods was to accept the fact that different individuals may learn in different ways, using different processes and therefore operate at different levels. Within §2.5.1 we discuss an integral representation used to support the initiative — the empty number line.

Even though the reasons for an initiative were similar, the response within England emphasised direct whole class teaching towards meeting satisfactory levels of achievement and an annual set of outcomes. One representation that it was considered would support the attainment of this initiative was the use of the number line and this forms the focus of §2.5.2.

2.5.1 The Empty Number Line

In 1990, Treffers & De Moor suggested:

> ...the idea of the old number line in a new format: the empty number line up to 100 as a more natural and transparent model than the hundred square.
>
> (cited in Beishuizen, 1999; p. 159)

Hall (1998) explains that the term "transparent" is used for those materials that show what the student is thinking to both the teacher and the student. The hundred square (or old number track model) consisted of ten rows with ten boxes in each row and consecutive numbers written within each box. It is a segmented number track, since it is spaces that are numbered, not a number line. However, this model seemed to confuse children when the boxes were presented without numbers so the notion of the empty number line (ENL) was developed. Gravemeijer (1994) explains that the empty number line was basically chosen because it is a linear representation that can reflect the counting aspect of number, unlike other representations such as Dienes blocks, which reflect the numerosity of number. The term numerosity is defined by Steffe, Richards, von Glaserfeld, Richards, & Cobb (1983, p. 41) as the "awareness of the 'amount of unit-slots' in the conceptual structure designated by a number word".

In 1997, the Dutch government proposed that the Freudenthal Institute should be involved in a project aimed at developing learning-teaching trajectories for primary school mathematics and a team was created to meet the needs of the project. This led to a movement towards realistic mathematics and resulted in the use of realistic textbooks (Beishuizen, 1999). It started with the Wiskobas project (Mathematics for Primary Schools), which developed into the recent Realistic Mathematics Education (RME) instructional theory (Treffers & Beishuizen, 1999). Realistic mathematics education was identified by Freudenthal (1973) as "mathematics education that is compatible with the idea of mathematics as a human activity" (cited in Gravemeijer, 1994, p. 445). The RME theory is also known as the "Theory of Progressive Mathematisation", that is the progression towards more formal problem-solving methods by raising the level of the students' activity (Beishuizen, 1997). The RME approach tries to overcome the separation that exists between informal and formal knowledge, by creating a path where students can reinvent formal mathematics by mathematising — organize from a mathematical point of view (Gravemeijer & Doorman, 1999).

2.5.2 England – Guidelines for Teachers

Concerns with levels of achievement in mathematics had also been a recurring theme during the late 1980's and the 1990's in the UK. An initial response was the introduction of a National Curriculum in Mathematics (DES, 1991) and, in response to the Education Reform Act of 1988 the implementation of Standard Attainment Tasks (SAT). However even with these initiatives, by 1998, only 59% of 11-year-olds achieved level 4 (the expected standard for their age) in mathematics.

In 1996, a two-year project, the National Numeracy Project carried out in selected schools, was designed to help the teaching of the National Curriculum in mathematics in primary schools (DfEE, 1998b). This project was proven to be a success mainly because it addressed and appeared to reduce underachievement in mathematics, amongst all groups of pupils (DfEE, 1998a).

Following the outcomes of the National Numeracy Project, a National Numeracy Strategy (NNS) (DfEE, 1999a) was implemented in English schools from September 1999. The document *Framework for Teaching Mathematics from Reception to Year 6* (1999a) which became an essential background source for this study, was closely associated with the National Curriculum for England and Wales (DfEE, 1999b) (NC).

The NNS provided paradigms of a typical (45-60 minute) lesson, consisting of:

- A 5-10 minute on mental oral work, for sharpening these skills.
- A 30-40 minute teaching activity (main part of the lesson) where teaching is followed by activities, which the pupils do either as an ability group, in pairs or individually.
- A 10-15 minute plenary where misconceptions are sorted, the day's lesson is summarised and linked to other work and pupils are given homework.

This style is not as formal as that of 'traditional teaching' where the lesson has homogeneity in style and the teacher is the main authority, discipline and source of knowledge. However, the fact that the style suggested by the NNS needs to follow a specific order and set of guidelines, is indicative of informal formality.

The NNS also provided guidance to supplement what must be taught in each Key Stage. Its purpose was to make sure that children become properly numerate, and defined numeracy as:

> ... a proficiency that involves a confidence and competence with numbers and measures. It requires understanding of the number system, a repertoire of computational skills and an inclination and ability to solve number problems in a variety of contexts. Numeracy also demands practical understanding of the ways in which information is gathered by counting and measuring, and is presented in graphs, diagrams, charts and tables.
>
> (DfEE, 1998b; p. 11)

Some skills contributing towards the formation of numerate pupils are (DfEE, 1999a, Introduction, p. 4):

- having a sense of the size of a number and where it fits into the number system.
- calculating accurately and efficiently, both mentally and with pencil and paper, drawing on a range of calculation strategies.
- being able to suggest suitable units for measuring, and make sensible estimates of measurements.

Among other issues, the NNS discussed ways to reduce the attainment gap, since there was a serious concern with standards and it suggested differentiated group work among children of different attainment groups.

2.5.3 The Number Line (within the National Numeracy Strategy)

Before the implementation of the NNS, other documents were circulated which promoted the use of the number line as a "key classroom resource" (QCA, 1998; DfEE, 1998a; DfEE, 1998b):

> There are several basic teaching resources that have been found to *help children master numeracy skills effectively*. ...every school should be equipped with these inexpensive resources... The key resources for each classroom are a board, *a large sized number line*, and a 100 square. *Resources for individual pupils are smaller number lines*, counters, ...
> (DfEE, 1998a, p. 23; Italics not in the original text)

In addition, since 1999 the NNS has produced a range of training materials and events delivered for primary school teachers by Local Education Authority Numeracy Consultants that were designed and written by NNS Regional Directors. One such item is a flash movie file, associated with the use of the number line, a representation that is used to support children's calculation strategies in addition and subtraction and their understanding of positive and negative whole numbers. This can be downloaded from:

http://www.standards.dfes.gov.uk/numeracy/publications/itps/number_line/

The use of the number line is a recurring theme within the National Numeracy Strategy. Appendices I and II (in both appendices the numbers 2.2, 3.2.8, etc. for example, stand for Section 2, page 2 and for Section 3, page 2, note 8 within the NNS) provide details of two themes associated with this study. These are extracted from within the NNS. Appendix I provides an overview of the number development and its association with the number line. Appendix II gives specific examples related with the use of the number line, number track, hundred square for procedures (counting, finding numbers, interpolating position, operations). Here a summary of these two Appendices is presented.

An examination of Appendix I indicates that within KS1 and KS2 we note the hierarchical development of the number system through use of the number track and the number line:

1. Introduction to number track (Reception)
2. Positioning numbers on a number track (Year 1)
3. Place numbers on a blank number line and hundred square (Years 1 and 2). (Note that there is no reference to what this is or what it might represent.)

4. Order whole numbers (naturals) and position them on a number track, number line and hundred square (Years 2 and 3). (Again it is worth noting that no clear reference distinction is made between the three and nor their individual properties.)
5. Order fractions (positive rationals), position them on a number line (Year 4)
6. Order decimals and position them on a number line (Year 5)
7. Order negative numbers (integers) and position them on a number line (Years 5 and 6)

More specifically within Appendix II we can see that the number line is implicitly used to support the development of understanding of the number system and associated with the development of counting skill to order numbers to 10 (Reception), to 20 (Y1), to 100 (Y2), fractions (Y2 onwards), decimals (Y4 onwards), negative numbers (Y4 onwards). Consequently, the following recurring themes are noted:

- *Counting and ordering* numbers: Reception (4.5, 4.12.b)[2], Y1 (5.2.a), Y2 (5.3.a), Y4 (6.28.a)

- Find or place the *missing numbers* on a number track: Reception (4.12.a, 4.12.c), Y1 (5.2.b, 5.10, 5.12), Y2 (5.3.b), Y3 (5.3) or a partially numbered number line Y2 (5.15.a), Y3 (5.15.a), Y4 (5.8, 6.14.a)

- *Estimating* the position of a particular number on a partially numbered or an empty number line *without the presence of an arrow*: Y2 (5.15.b, 5.23), Y3 (5.15.b), Y4 (6.14.b, 6.28.b)

- *Estimating* the position of a particular number on a partially numbered or an empty number line *with the presence of an arrow*: Reception (4.9), Y2 (5.11, 5.17), Y3 (5.11, 5.17), Y4 (6.8, 6.10), Y5 (6.11, 6.15), Y6 (6.11)

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[2] Reception (4.5, 4.12.b) is a reference to the Reception year, within section 4, page 5 and section 4, page 12, note b of the NNS.

- Perform *operations* using a number track: Reception (4.15) or the empty number line Y3 (5.43, 5.45)

2.5.4 The Number Line — Procedural Use

The picture that has emerged is that despite its frequent appearance and plethora of applications within the NNS, the use of the number within the NNS tends to emphasise its procedural use as a tool rather than its conceptual structure. The following statements would seem to support this view:

> *Make frequent use of a number line*, 100 square, number apparatus, pictures, diagrams,... and games and puzzles where *the rules are picked up quickly by watching a demonstration.* (NNS, Introduction; p. 21; Italics not in the original text)

> **Demonstrating** showing, describing and modelling mathematics using appropriate resources and visual displays:... *demonstrating on a number line how to add on by bridging through 10...* (NNS, Introduction; p. 12; Italics not in the original text)

Foster (2001, see also §2.3.2) has indicated the shortcomings of the use of demonstration whilst it may be the limited emphasis on a developmental approach that the QCA (1998) reported that "The number line sequence in the final question of the test was beyond the understanding of all but 18% of children" (QCA, 1998, p. 23). This question required that children gave the number that was pinpointed by an arrow. It would seem that in an attempt to raise children's levels of awareness and achieve an improvement in the proportion of correct responses within the NNS similar activities to this one are addressed [See Appendix II and more specifically Reception (4.9), Y2 (5.11, 5.17), Y3 (5.11, 5.17), Y4 (6.8, 6.10), Y5 (6.11, 6.15), Y6 (6.11)].

Explicit reference to conceptual knowledge associated with the form and use of the number line appears to be omitted from within the NNS. The number line is not explicitly defined, but is seen as:

> ... a means of showing how the process of counting forward and then back works. It can also be a useful way of getting children to visualise similar examples when working mentally. (QCA, 1999a; p. 31)

the final objective is for "children to calculate mentally" (DfEE, 1998; p. 30).

Thus it could be claimed that the inclusion of the number line within the curriculum is as a means to and end, its use as a tool, but achieving the end appears to ignore what it is the children are using and, beyond the more obvious perceptual differences, also ignores the conceptual differences between number tracks and the number line.

> Beside a board, each classroom should have a large, long number line for teaching purposes, perhaps below the board, and at a level at which you and the children can touch it. A "*washing line*" of numbers strung across the room, and which can be added to or altered, is useful. Provide *table-top number lines*, marked and unmarked, for individual use. For Reception and Year 1, *number tracks* with the spaces numbered to 20, rather than number lines with the points numbered, are helpful, including those made from carpet tiles. You could also have a *floor "snake"* for children to move along in corridors, the hall and the playground. For Year 2, lines need to extend to 100; by Year 4 they should include negative numbers; Years 5 and 6 need *marked and unmarked lines* on which decimals and fractions can be placed.
>
> (NNS, Introduction; p. 29-30; Italics not in the original text)

Each of these representations can have a use but collectively, without the identification of the conceptual differences between them, they are a potential source of confusion not only in naming, but in later recognising the potential, which is unique to the number line. The usual washing line that is displayed in the classroom is an analogue of a number track. The difference between it an a number line does not lie simply in the perceptual sense that one has the "spaces numbered" and the other has the "points numbered" but in the conceptual sense identified by Skemp:

> On the number line, numbers are represented by points, not spaces; and operations such as addition and subtraction are represented by movements over intervals on the line, to the right for addition and to the left for subtraction. The concept of a unit interval thus replaces that of a unit object. Also, the number line starts at 0, not at 1. For the counting numbers, and *all positive numbers*, we use only the *right-hand half of the number line, starting at zero and extending indefinitely to the right. For positive and negative numbers we still use 0 for the origin, but now the number line extends indefinitely to the right (positive numbers) and left (negative numbers).*
>
> (Skemp, 1989; pp. 139-141; Italics not in the original text)

There is no 'zero' on a number track and neither is conceptual understanding associated with it as a key point of reference, the start of the chosen interval and the repeated transformations that provide the continuity which is the source of the number line's strength as a representation of the number system. Nevertheless, within curriculum related documentation the number line is treated as a discrete model, not only as an alternative version of the number track but also as a fragmented facet of the number system. For example within QCA (1999a, p. 53) number system understanding is encouraged by the presentation of three discrete number lines: one line for fractions, another for decimals and another for percentages. Though Herbst (1997) uses a similar representation, presenting fragmented lines of sets of numbers, he eventually brings them together to form a single line to represent the number system. This emphasis does not appear to be explicit in NNS documentation.

The mediator between the National Numeracy Strategy's guidelines and the pupil is the teacher. The teacher's knowledge and beliefs have a great impact on their classroom teaching and consequently on the children's construction of knowledge from their experience within a lesson. Therefore, the issue of teachers' subject knowledge or subject-matter understanding is raised within the next section.

2.5.5 Teachers' Subject Knowledge and Beliefs

Shulman (1986) defines subject matter (content) knowledge as "the amount and organisation of knowledge per se in the mind of the teacher" (p. 9) and distinguishes between the aspects of knowing "that" and knowing "why". Aubrey (1994) suggests that every teacher has different subject knowledge and personal beliefs about teaching and learning, which are factors affecting their work in classroom; and in order for teaching to be effective, conceptual understanding of knowledge is essential. However, Aubrey (1994) states that:

> The provision of a rich and deep subject knowledge may, however, lead to the coverage of less subject content which the National Curriculum does not allow! (p. 192)

Brown, Askew, Baker, Denvir & Millett (1998) argue that in its intention of aiding teachers through the implementation of recommended practices, the National

Numeracy Strategy neglects considering teachers as individuals. In an investigation on teachers' beliefs about the teaching and learning of numeracy as well as their interaction with pupils in class, teachers unanimously expressed positive views about the NNS, in terms of helping them to gain more knowledge about ways of teaching and the curriculum. However, the changes that have taken place since the implementation of the NNS are not that deep and teachers have not really altered the core of their teaching (Askew, Brown, Rhodes, Wiliam & Johnson, 1997a,b; Brown, Askew, Millet & Rhodes, 2003).

Askew, Brown, Rhodes, Wiliam & Johnson (1997c) distinguish between three orientations towards mathematics teaching beliefs. Connectionist teachers value both pupils' methods and teaching strategies, as well as establish links with mathematical ideas. Teachers who portray "transmission" beliefs prioritise teaching over learning and consider mathematics to be a collection of routines and procedures. Teachers with "discovery" beliefs prioritise learning over teaching and consider mathematics as being discovered by the learner. Askew *et al.* suggest that teachers may combine characteristics of two or more of these orientations, but those teachers with stronger connectionist orientations are more likely to have greater gains in terms of learning, mainly because they acknowledge the role of both the teacher and the pupils in lessons. Effective teachers can be distinguished from less effective teachers in terms of increased fluency in discussing conceptual connections in terms of classroom practice and less effective teachers may express a more procedural rather than conceptual personal subject knowledge (Askew *et al.*, 1997a).

Though not a central issue within this study, subject matter knowledge is of course relevant in the context of the teachers' awareness of conceptual issues associated with understanding the nature of the number line, particularly its underlying continuity. This is a feature examined within Chapter 5. Additionally, and perhaps giving relevance to Aubrey's view, is the way that teachers use the number line within their classrooms and what it is that they see as important to communicate to the children they teach. This therefore leads to further issues for the study:

What is the teachers' perception of the number line and how do they use it as a pedagogic tool?

2.5.6 Summary of Section 2.5

The number line is clearly a very different representation to the number track (Skemp, 1989). However, evidence drawn from within the NNS (DfEE, 1999a) suggests that there is no explicit evidence indicating a conceptual difference between the two. The success of the number line in Dutch schools is mainly attributed to strong research foundation on the particular model (Klein *et al.*, 1998; Beishuizen, 1999). This success may have contributed towards the implementation of the number line within the English curriculum, where it seems implicit that similar results were and are expected through its use. However, within the NNS, there appears to be an implicit assumption that those who use the number line are fully aware of not only its use as a tool but also of the conceptual understanding that underscores this use. Teachers find themselves in between the guidelines and learning, therefore their teaching and subject knowledge become crucial to what is to be learned. The consequence of the difference between using the number line as a tool and having conceptual understanding of its nature become a central issue for this study.

2.6 Concluding Remarks

As we have seen from the review of literature, several issues are raised when instructional materials are used for the teaching and learning of mathematics. For example, each child constructs meaning from a common representation based upon their unique conceptual structure (Fuson *et al.*, 1997) and consequently may interpret the representation in a different way to that intended (Cobb *et al.*, 1992). Though they may learn how to use the representation, they may not be conceptualising the mathematical meaning behind it (Dufour-Janvier *et al.*, 1987).

This study focuses on a particular representation that is the number line, and investigates its teaching and learning within English classrooms. It is an abstract and sophisticated representation evolving from plane geometry and though defined in various forms by Williams & Shuard (1970), Skemp (1989) and Herbst (1997) there appears to be no definition or explanation of meaning within the NNS and its supplements (DfEE, 1998a; DfEE, 1998b; DfEE, 1999a; DfEE, 1999b; DES, 1991). Furthermore, its frequent appearance and extensive procedural use within the NNS and its supplements raise issues, as to whether the guidelines and expectations suggested within the NNS which emphasise the use of the number line as a tool rather than the representation of sophisticated ideas contribute towards a procedural orientation rather than conceptual way of teaching and learning.

This thesis examines the individual child's perception, understanding and use of the number line and what it is they construe from those lessons where, notwithstanding the use of symbolism, it is the dominant representation. This is contextualised with a consideration to the teacher's perception and understanding of the number line although a deeper investigation of this proved to be difficult.

We now turn to consider the preliminary studies, which helped to crystallise the main study's research questions and have contributed towards its design.

Chapter 3: Preliminary Studies

3.1 Introduction

An 'exploratory' study and a 'pilot' study provided the basis for the development of the methods applied within the main study. Consequently, the outcome from these will be considered before formal consideration of methodological considerations. The 'exploratory' study was developed to gain a clearer perception of the way a variety of individuals understand the number line and to examine the best way of developing a framework to resolve the key research questions. The 'pilot' study was developed to consider the way that English pupils would react to issues emerging from the 'exploratory' study. Consequently, the pilot study is an integral part of the development of a suitable methodology. Both it and the exploratory study not only contribute to the final methodology from a data collection perspective but also from the perspective of data analysis. To ease general discussion these earlier studies will be referred to as preliminary studies.

3.1.1 The Aims

Though generally applied at a relatively informal level, in that it was responses of respondents that guided the framework of the exploratory study, its main purposes were to:

1. Investigate the use of research instruments through which an understanding and interpretation of the number line could be discerned

2. Establish a sense of the phenomena that may be associated with the individual's understanding and interpretation of the number line and its contribution towards the development of number knowledge

3. Provide a foundation for a pilot study within an English School

4. Provide the researcher with the opportunity to develop her interview technique

3.1.2 The Questions

To guide the preliminary studies four main clusters of questions were identified:

(i) Those associated with identifying respondents' notions of the word 'number'. These questions were used to establish whether or not a number line was an intrinsic part of the respondent's thinking about number.

(ii) Those which examined the phenomena individuals choose to mention when asked to indicate "What is a number line?"

(iii) Those associated with the inclusion or otherwise of additional information that may illustrate whether or not respondents recognised the underlying features of replicated unit and continuity.

(iv) The approach and associated accuracy with which specific numbers may be estimated. These items would provide a sense of the inherent features that pupils recognised within the number line.

The research instruments for the preliminary studies were semi-structured interviews (see Appendix III). Based upon the response of each individual, supplementary questions were asked to clarify and deepen the interviewee's understanding of the initial responses (see §4.3.2).

3.1.3 The Sample

A diverse sample was identified for the preliminary studies to gain a sense of the way in which the number line was interpreted, understood and described by those for whom it was both a familiar and an unfamiliar representation. Thus, the sample was essentially opportunistic and drawn from groups within Cyprus (Teacher training undergraduates and primary school children who had not had experience of the number line) and England (PhD students). The main reasons for considering Cypriot pupils in the first instance was partly because it provided the researcher with an opportunity to carry out a qualitative investigation in her first language and secondly it enabled some conclusions to be drawn about respondents' intuitive conceptions of the number line. The general responses from this diverse sample provided the basis for a more focussed investigation of English primary pupils' conceptions of the number line. Investigation with this group formed the basis through which the methodology to be used within the main study would be reported (See Chapter 4). Consequently, this chapter is presented in two parts. Initially it presents an overview of a diverse range of respondents' understanding of the number line and then it focuses on issues and responses from these responses to consider in more depth the English primary pupils' understanding of a number line.

3.2 The Exploratory Investigation

The respondents in the exploratory investigation were Cypriot Teacher Trainees (N=7), primary school pupils within Year 3 (median age 7.5, N=9) and graduate students (N=7) within the University of Warwick. Initially the questions presented to these various groups focused on their perceptions of number and particularly on the perception of the numbers one to one hundred but as interviews with various groups progressed, and as a result of the evidence obtained from the initial interviews, additional questions that focussed on perceptions of the number line and the addition of questions associated with estimating position were included.

3.2.1 Conceptions of Number

The initial question asked of all respondents was "What comes to mind when you hear the word 'number'?" (Appendix III).

The Cypriot children's responses to the question included responses that were associated with things, areas where numbers were applied, isolated examples of numbers or multiple examples and in one instance associated with actions with numbers.

Two of the responses began by giving what may be described as a "general" response (Pitta & Gray, 1999) but further questioning established the nature of meaning derived from this response. Child 3, for example initially responded by indicating "Number" but when asked to clarify gave random examples. Child 1 gave a similar initial response, but on further questioning simply inquired whether she should tell the interviewer the numbers. Child 4 simply responded "Digit" but was unable to clarify what this meant.

The word number was associated with 'mathematics' (Child 8) and with things and personal episodes (Child 2), the latter indicating:

> Animals, the people... The trees, the plants...the cars... because these are a lot, it's like counting them.... Because I'm counting them. Sometimes, because I get bored in my house. I count them. (Child 2)

Only two Children, 7 and 9, immediately suggested multiple examples, the latter indicating that she saw:

> Different numbers... in a row ... the ten and the twenty bigger than the others (Child 9)

Child 9 (the only one of the children) expanded on their initial comments to provide a comment that could be inferred as a reference to something that could be described as analogous to the number line. Her use of the word "row" and specific reference to the ten and the twenty does carry some implication that she is talking about an image associated with a number line but this could not be clarified further.

Five of the eight undergraduates who provided a specific number as a response associated this number with a particular colour:

> Nine in my own handwriting. It's black. Then the ten came. Maybe because I am nineteen years old. I also have the number nineteen in my user code to enter the university computers. When they tell you the word "number", the numbers from one to ten come to your mind. (Undergraduate 3)

> Three. Blue. In my own handwriting. Alone. Clear. There is a three in empty space. My lucky number. (Undergraduate 5)

> Eight. As it is. Red. Two circles, one smaller than the other. It was there. Still. In my own handwriting. Clear. Something lifeless. (Undergraduate 7)

> The [symbol] two. In my own handwriting. Red. Came and left. About thirty centimetres above my head. As if looking at a blackboard in front of me. (Undergraduate 2)

The responses form these students, possessed similar qualities in that they were descriptive: they associated the notion of number with one specific number that seemed to carry the same qualities, in that they identified colour, as those described by Seron, Pesenti, Noel, Deloche & Cornet (1992).

However, two of the undergraduates provided responses that were more sophisticated:

> Different numbers. Two, four, six, eight. The symbol comes. The meaning of number. Big numbers, series, progressions. Just the symbols. (Undergraduate 6)

> Seven goats, area of a square, sides of a hexagon, root pi ($\sqrt{\pi}$), combination of different squares in a hundred square. The numbers I might play in "joker" tonight. Nine, twelve, seventeen, twenty-two, thirty-seven, forty-seven. Seven different combinations to get the seven. One and one and two and three equals seven. One and two and four equals seven. .... (Undergraduate 1)

These two responses reflected the sort of comments Pitta (1998) categorised as generic or proceptual. The first provides the sense that the notion of number could take Undergraduate 6 in a variety of directions whilst the second, though it also suggests that Undergraduate 1 could go in a variety of directions, also suggests the particular strength of symbolism as representative of number or as an expression of procept (§2.2.4) by giving examples of different procedures that provide 7 and the inclusion of the symbol $\sqrt{\pi}$.

Amongst the Warwick graduate students, whist there was also evidence suggesting a focus upon specific examples that were associated with colour, there was also evidence from one of the seven students that these numbers were in the form of a number line:

> A series of numbers on a line. Still numbers. Black. I picture the line on a white background, like a sheet of paper. Vertical line because I used to make bullet points (writes 0, 1, 2, 3 until 10 one under the other). (Graduate 3)

Another provided an indication that his perception of number was associated with a number square although he referred to it as a number line:

> I see numbers. Number line, increases left to right. It keeps going. It's like a matrix. Something like up to one hundred. From one to ten, ten to twenty, ... They are in front of me on my eye level. (Graduate 4)

Overall, the evidence suggests that the children give specific responses that reflect simplicity, whilst both the undergraduates and the graduates could elaborate on their responses to provide evidence of the indications given by Seron, Pesenti, Noel, Deloche & Cornet (1992) and the general nature of their "number forms" (Galton, 1880, 1907). However, across the full sample the notion of number line was only used twice. In both instances, these were by graduate students but one of these gave an outlined description of a hundred square.

3.2.2 Number Line Conceptions

It was in an attempt to direct thinking towards a more frequent expression of the term 'number' that a second question "What comes to mind when you think of the numbers 1 to 100?" (Appendix III) was asked of each respondent.

Only two of the children provided any response indicating that they thought of the numbers 1 to 100 in a form analogous to a number line, but the term number line was not used by either. Although, for example, Child 4 initially began to write the numbers one under the other starting from 1 she also indicated:

> Yes, but I think of them like this [initially making horizontal left to right movements, with her right hand and she then proceeded to write a sequence of numbers from one in a row] (Child 7)

Children 3 and 6 explicitly mentioned the notion of 'row' whilst Child 8 indicated the notion of 'line':

> This line comes. It comes for all the numbers. I always see it. (Child 8)

The notions of 'row' or 'order' were used by half of the Cypriot children. The explanation of Child 7 was accompanied by left to right hand movements. For Child 1, thinking of the numbers 1 to 100, invoked the action of 'counting' whilst Child 5 indicated that when he thought of the numbers 1 to 100 he 'always felt like counting'. Child 8 indicated that thinking of the numbers 1 to 100 makes him sing a rhyme associated with the multiples of five because they use this in their games of hide and seek.

The results from the children again provided no explicit reference to a 'number line' although there are several instances where concepts that could be associated with a number line, order and row, are used.

Four of the undergraduate students again associated strong specific images with their explanation of what came to mind with the numbers 1 to 100. Undergraduate 7 suggested that she saw the numbers in a row with the tens more obvious and of different colours. Two others indicated that the numbers were in a row but no actual line was seen:

> No line. Black on a white background. I clearly see the one and the one hundred and imagine dots in between. (Undergraduate 3)

> [Associated with left to right hand movements] Here is the one (Left) and here the one hundred (Right), with a chunk of numbers in between. Black. Plain. On eye level. They are in a row, but there is no actual line. (Undergraduate 7)

A fourth explicitly mentioned 'line':

> A line. A line without the numbers. Not very clear. It came and left. As if it was swinging. Black. They came automatically. On eye level. (Undergraduate 4)

Of the remaining students, one simply indicated that he thought of a line whilst the other two either thought of row or a random selection of numbers.

The numbers 1 to 100 were associated with the notion of line by all of the graduate students, but the unit (U) was frequently compressed, as the range of numbers grew larger:

> Like...mm... it begins one, two, three like very clear numbers and then ... coming to one hundred it gets blurred, and ... like the numbers close together, less clear than the first bit. (hand movement from left to right) (Graduate 5)

Clearly this particular student is describing a 'number form' that fits many of the descriptions supplied by Galton (§2.2.6) that imply that as the numbers move away from 1 the size of the unit space (U) decreases so that the numbers become less clear. An alternative view of this mechanism is highlighting the tens only:

> I think of them as being on a straight line in groups of ten, so.... And you can clearly see the tens. (Graduate 6)

The responses of the full sample to the invitation to consider "What comes to mind when you think of the word 'number' and the numbers 1 to 100?" suggests that the notion of number line does not figure directly in the children's perceptions of number and figures only in relatively isolated instances amongst the undergraduate and graduate students. Of the 23 respondents to this phase of the study, just less than 40% provided references to the use of a number line or to a row of numbers without explicit reference to the line. This proportion was biased towards the elder respondents with almost equal proportions making reference to notions of 'number line', 'line' or 'row'. Only 2 of the graduate students used the term 'number line', whilst none of the children did, although 5 of these made reference to notions of 'line' or 'row'.

The quality of the responses given by the respondents also indicated some differences. Although these were largely associated with the presentation of specific examples, some of these evoked references to colour and position in the mind, others were deemed to be associated with past experience, in that they associated a verbal symbol with objects (Child 2), whilst the difference between these types of articulation and those given by Undergraduates 1 and 6 to the word 'number' are striking. On the one hand, we see articulations that are essentially descriptive in that they provide specific examples, whilst on the other we see examples of responses that are relational in that they broaden the scope of the initial comments to embrace a wide variety of associated concepts. These two features were worthy of further consideration as the exploratory study progressed.

It had been hoped that without explicit reference to the notion of 'number line' by the researcher, there would have been more evidence of reference to it or associated terms such as 'row' or 'line' by the respondents to have enabled the interview procedures to probe these points further from there. However, this was not the case so the interview procedure was amended to include direct reference either verbally or visually to the number line.

3.2.3 Visually Presented Number Line

Responses to the items identified above guided the preparation of one other key question associated with respondents' interpretation of a 0 to 100 segment of a number line partitioned into tens (see Appendix III, Q.3). Each respondent was presented a visual representation of such a line and was invited to indicate what they thought it was. After their responses, the children and the undergraduates were then asked to consider whether or not there was space for other numbers.

Only one of the children (7) identified the representation as a number line. Child 2 considered that it was an "Abacus" whilst the others variously described what it showed:

> A line that has lines on it. (Child 3)
>
> It's up to one hundred. Ten, twenty, thirty, forty,... one hundred. (Child 5)
>
> Instead of one, two, three, four, ... one hundred, we put intervals of ten-ten. (Child 6)

Child 9 simply indicated that it had something to do with arithmetic and numbers.

4 of the 7 undergraduates identified the line as a number line that went "10-10" (Undergraduates 2 and 3) and with a beginning and an end (Undergraduate 4). One (Undergraduate 7) identified it as a ruler whilst one (Undergraduate 6) identified it as an "arithmetic series". The final student indicated that:

> It's a representation of the whole numbers by ten, from zero to one hundred inclusive. I don't know if there are numbers in between, as there are no intervals between ten and twenty for example. (Undergraduate 1)

4 of the 6 graduate students recognised the presentation as a number line the other 2 suggested that it was like a ruler.

To gain further understanding of the interpretations of the representation, the children were asked if there was space for other numbers and whether or not they could estimate the positions of 79 and 3 that were not actually indicated on the line.

Two of the children indicated that there was no space for other numbers:

> ... because the line has finished. (Child 4)

Child 1 suggested that if the line was extended to the right there would be more space. Two of the children (Child 3 and Child 7) indicated that a zero could be added whilst Children 5 and 9 indicated that one other number (21 and 11 respectively) could be added.

The general picture that emerged from this question was confirmed in the children's attempts to estimate the position of 79 and 3. Child 6, who did not respond to the previous question, was unable to respond to this one, whilst children 1, 3 and 4 who suggested that no number could be identified between the tens, also indicated that 79 and 3 were not on the line:

> Seventy-nine is somewhere behind eighty. (Child 1)
>
> Because it goes in tens. They are not on the line. (Child 3)
>
> There is no three. There is a thirty. (Child 4)

The undergraduate students were asked whether or not there was space for other numbers. Initially, Undergraduate 4 indicated that there was not, whilst three (Undergraduates 2, 3 and 7) indicated that there was, if the line was extended. Undergraduate 2 did however, qualify his comment to include marking the halves:

> Yes! If I extend the line, or if I mark the halves (Undergraduate 2)

After his initial response, Undergraduate 4 suggested the inclusion of the halves and the decimals whilst Undergraduate 1 suggested that he could put anything on it:

> I created the system. I create the line. (Undergraduate 1)

This comment was later refined to include an implied reference to continuity:

> I can separate the line in another way as well. By writing rational and irrational numbers between zero and one. (Undergraduate 1)

Though there was no other implied reference to continuity, the infinite nature of the line was considered by 3 undergraduates:

> Not that it matters....since it goes to infinity. (Undergraduate 6)

> If it is a line it is infinite. If it is a linear section then it may start from A and end at B. If I want to count on a number line I would definitely place zero at A and it depends for B. The last number one hundred, one million. (Undergraduate 3)

> The beginning and the end are subjective. The line could start either from zero or one, two, three,... or five, ten, fifteen... I would prefer five, ten, fifteen,... because five is my lucky number. (Undergraduate 4)

When the graduate students were asked to define a number line, their responses ranged from the descriptive:

> Well, a number line would be ... a line where a point on the line would represent a number and equal distances on the line would represent equal differences between the numbers. So it's just a geometrical representation of numbers. (Graduate 6)

through to a recognition of its association with magnitude:

> It is a line that represents the number system and....it's ... I would make reference to it ... when I think of numbers as magnitudes. (Graduate 4)

Though none provided a clear definition of it, its abstract nature was identified:

> The representation of the numbers. I mean, it is something very abstract that teachers use in schools to show the kids the numbers. (Graduate 4)

3.2.4 Estimating Numbers on a Number Line

The responses of the sample to issues of whether or not additional numbers could be included within the 0 to 100 line, generated interest in the ability to estimate the position of numbers. Each pupil was asked to estimate the position of particular numbers on a number line marked 0 to 100 (see Appendix III, Q.4). In a replication of questions asked by Doritou (2001) all respondents were asked to mark the position of the numbers 93, 45, 12, 5, 75, 3, 7, 25, 97, 95, 88, 55. These numbers represented pairs that were equidistant from the extreme points of the line (see §4.3.3).

Figure 3.1 illustrates the distribution of the mean errors of each of the three samples; children, undergraduates and the graduates. The mean absolute error was calculated by considering the mean of the absolute difference between the estimate and the actual number for each respondent within each category.

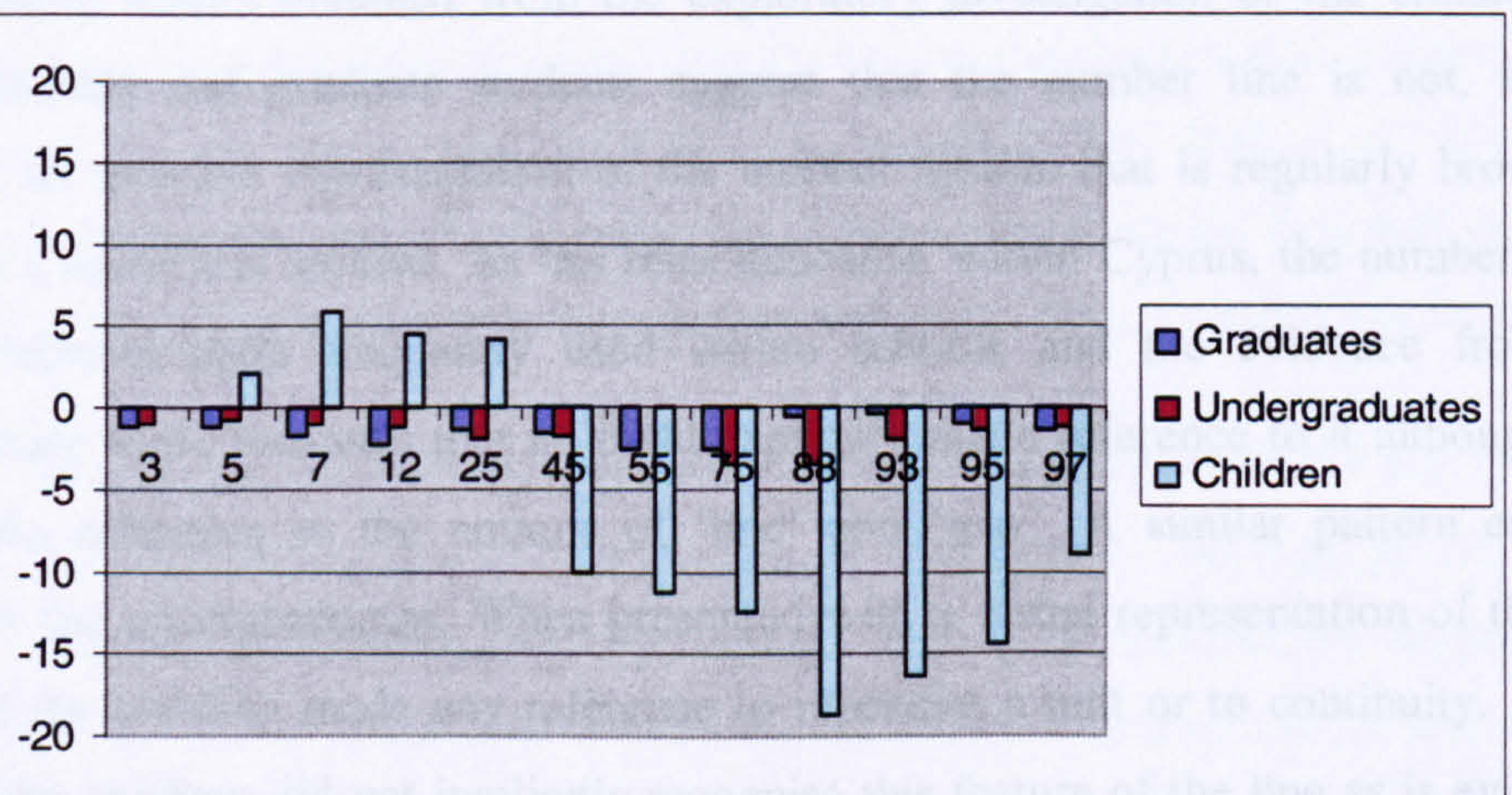

Figure 3.1: Mean errors for interpolated numbers on a number line

Several features emerge from this figure:

- The mean of the absolute error for each of the children's estimates, apart from their estimate for number 3, is greater than the mean of the absolute errors of the undergraduates and the graduate students.

- The estimates of the undergraduates and the graduates is better than that of the children.
- The children over-estimate the magnitude of numbers below 45 and underestimate the value of the numbers 45 and above.
- Even though the numbers to be estimated are equidistant from the end points of the line, children tend to make a much larger error when estimating numbers over 50 than they do when estimating numbers under 50. The maximum extent of underestimation can be greater than that of overestimation.
- Errors for numbers below 50 appear to peak at 12 for undergraduates and 7 for the children. Errors associated with numbers above 50 peak at 88 for both groups.

The general results obtained from the exploratory investigation of the children, the undergraduate and graduate students suggest that the number line is not, for this sample, an intuitive representation of the number system that is regularly brought to mind in a numerical context. As has been indicated, within Cyprus, the number line is not a representation frequently used within schools and the evidence from this exploratory study indicates that no child explicitly made reference to it although they did make reference to the notions of 'line' and 'row'. A similar pattern emerged amongst the undergraduates. When presented with a visual representation of the line, none of the children made any reference to repeating a unit or to continuity. Almost half of the children did not implicitly recognise this feature of the line as is evidenced by their indications that the numbers 79 and 3 could not be placed on the line. Only one Undergraduate (7) did not clarify her view that additional numbers could only be placed on the line if it was extended.

It was therefore something of a contradiction that all respondents later attempted to estimate the position of a variety of numbers on a number line segment from 0 to 100, but the results, particularly those of the children, warranted further consideration through the pilot study.

3.3 Pilot Study: English Children

The outcomes of the exploratory study suggested that the three key issues should be considered with children within an English school:

- What comes to mind when they hear the word 'number'?

  In contrast to all of the previous sample, these children had received mathematics lessons guided by the principles and outcomes of the NNS (See §2.5.3). Therefore, this question was an attempt to consider whether the children's notion of number included an embodiment of the number line. Thus confirming the results of Galton (1880, 1907) although he suggested that only about 1 in 20 of children displayed such an embodiment, and the conjecture of Dehaene (1997).

- What does the word 'number line' mean to the children?

  Since they had received extensive experience using the number line, a question that considered this issue was included to establish the degree of conceptual understanding that the children possessed about a representation they had used and, because of the curricular recommendations within the NNS, would continue to use.

- The errors associated with the estimates of the pairs of numbers by the groups within the exploratory study provided an interesting distribution. Although at the time of preparing the pilot study the hypothesised models of distribution suggested by Siegler *et al.* (2003, 2004, 2005) were not known by the researcher, it was felt that the distribution identified from a group of children who had not experienced the number line was worth considering against a group of children who had experienced it.

Since the data within the exploratory study was derived from questionnaire and interview and the analysis of the results had suggested general trends in articulation of ideas that suggested clarification through episodes or through particular examples, it was considered appropriate to continue with interviews as one form of data collection. However, since the time to carry out these interviews was finite, questionnaires had also been used. These had proven to provide a wealth of data for descriptive analysis and so the pattern of interview supported by questionnaire was continued.

14 children within the Year 4 class (median age 8.5) of a school within the English Midlands were randomly selected for a series of individual interviews to consider their perceptions of number and the number line. Guided by the experiences within the exploratory study, each child was interviewed individually up to five times, depending on the quality of the information obtained each time. Each interview would last from ten to maximum twenty minutes. These took place during a period of three-four weeks in the school.

3.3.1 Conceptions of Number

The responses of 7 of the 14 children to the question "What comes to mind when you think of the word 'number'?" were associated with the invocation of some form of action:

> It means that you have got different numbers and you just start counting them. (Y4.6[3])
>
> Counting up on my fingers and working it out. (Y4.14)
>
> Adding, subtraction... maths really. Because number has to do with maths. (Y4.3)

Some responses elaborated the actions with examples using specific numbers:

> It is something... when you're counting, you count up numbers, say one, two, three, four, five, six, seven, eight, nine, ten... they're numbers and those all different kinds of numbers all over like... em... two... you've got two eyes, and then you've got ten toes, so it's like different sorts of numbers all over there. (Y4.10)

There was also evidence of responses that were similar to those of the Cypriot children in that they were more general, for example, simply responding with the word "Number". The interesting feature was that, unlike the Cypriot children or the

---

[3] Y4.6 stands for Child 6, Year 4

undergraduates, no child gave a response by illustrating with a specific number. However, one child did begin to illustrate the variety of ways that numbers can be used:

> Well... a number means to me... it's.... it's.... A number! A number is... when you add something up, you can add a number... like ten times ten... you can add ten add ten and get twenty. A number can mean anything. A number can mean your age... ten, nine... and a number can mean.... a number for... a number of children. Sixty children. That's what number can mean. It means many things and any stuff. (Y4.4)

One child (8), responded by indicating that a number line came to mind when she though of 'number' but when asked to indicate what a number line was she provided a descriptive response of a specific representation more associated with the notion of ruler:

> It's... a bit of plastic... a line of plastic is a number line.... and made of centimetres and metres. (Y4.8)

3.3.2 Number Line Conceptions

The responses to the question "What is a number line?" could be categorised into two main types – either descriptions of some features or an association with some sort of action. Only one other child (2) gave a simple descriptive comment without additional information:

> It's got numbers on. (Y4.2)

In two other instances, what started with a description of a specific line was supported by reference to the use of the line:

> ... you just use the number line to count-up. (Y4.11)

> It's got all the numbers from one to one hundred (hand movement)... The number line can help you because if you're stuck on a sum and it's between twenty-two and fifty... you're stuck on it. You can look at the number line and then it... you get... draw a picture of the number line in your brain and it sort of like helps you because... number lines can be very useful and they can help me and they're quite useful to me. (Y4.10)

10 of the 14 children made reference to what can be done with a number line. Some of the responses were directly associated with evidence of episodes derived from classroom activity:

> When it goes up any number... you have got to work out what the missing number is... You've got a line and you do little hops like frogs do and then you add on each time until you get the answer... you count on them like we did in Year 1 and Year 2. (Y4.7)

Others gave examples of processes that were frequently associated with specific examples:

> You can make... you can count up in times tables... like three, six, nine, twelve... (Y4.5)

> A number line is... it's easy when using some things and you got to do additions... say... twenty-four add sixty-two... and you needed help... Go onto the number line if you wanted. (Y4.4)

> If there was a hard question you could use the number line to jump places, so... two hundred and seventy-three to three hundred and eighteen. (Y4.3)

This extensive emphasis upon things that can be done with a number line, in other words strong recognition of the number line as a tool, was not tempered by any sense of the underlying structure of the line. Perhaps the strongest indication of the emphasis placed upon the use of the line was identified by Child 14, who indicated that a number line was "Partition".

Such indications guided the structure of the main study for this thesis in that they suggested that a sharper focus should be placed on the way that teachers use the number line within the classroom and the interpretations that children made of that use (see Chapter 6).

3.3.3 Estimating Numbers

As with the respondents within the exploratory study, the children within the pilot study were asked to estimate the magnitude of the numbers 93, 45, 12, 5, 75, 3, 7, 25, 97, 95, 88, 55. As was seen in (§2.5.3) a similar type of activity is a recurring theme within the NNS, however, an analysis of the differences between the means of the children's estimation for each number and the number, provided a distribution that was remarkably similar to that of the Cypriot children in that:

- the numbers below 45 were overestimated

- numbers 45 and greater were under-estimated

Although the magnitude of the differences identified amongst the English pupils was less than that of the Cypriot children, the similarity in the trends suggests that this phenomena was worthy of further consideration in the main study. It is a particularly interesting feature, because not withstanding the fact that the estimations of the English pupils were closer to the actual magnitudes than those of the Cypriot children — for example for number 7 the English pupils over-estimated by a magnitude of 1 whereas the Cypriot pupils overestimated by a magnitude of 6, for 45 the under-estimates were 4 and 10 respectively and for 93 they were 7 and 16 — the pattern of the distributions was similar even though, the Cypriot pupils had not been introduced to the number line as a representation of the number system nor, unlike the English children, as a tool for addition and subtraction.

During their interviews, the English children were asked to indicate how they arrived at a particular estimate. All of the children used either an accumulating strategy — counting in ones from zero — for at least some of the estimates, and frequently, particularly if the number to be estimated was very close to 100, they used a decrementing strategy — counting down from 100.

> Coz zero, one, two, three, four five... six, seven... six would go there, seven... eight... nine... that should be ten. Eleven... twelve.
>
> (Y4.14, estimating 12[4], identified position of 23[5])

[4] Asked to pinpoint the position of number 12 on a 0 to 100 line
[5] The result of the child's estimation was to actually identify the position of number 23 rather than 12

> One, two, three, four, five, six, seven, eight, nine, ten, eleven, twelve, thirteen, fourteen, fifteen, sixteen, seventeen, ... forty-one, forty-two, forty-three, forty-four, forty-five.
>
> (Y4.8, estimating 45, identified 46)

> ... Because it is very near the end of the hundred, but it's not there. And there would be about ninety-nine and then in here it would be ninety-eight... and you just go ninety-seven.
>
> (Y4.10 estimating 97, identified 95)

In one instance, the accumulation strategy used by a child involved initially sequencing in fives and then sequencing in tens but the estimates of the unit gaps were so large that the child's estimate of 55 finished almost at 100 without any recognition from the child of the difference in magnitude between the two nor the weakness in his approach. As he did his leaps of five, he indicated (marking the positions with arrows under them):

> That would be five... that would be ten ... that would be fifteen... When I get to fifty something good happens. Then it goes up in tens. (Y4.1, estimating 55)

Unfortunately as he went up in tens there was little room left on the number line so marks for 60, 70, 80 and 90 were squashed up near 100.

In their use of the accumulation approach, children used a variety of markers. Child 14 simply marked by pointing to positions to leave no tangible evidence of how an estimate was arrived at. Others left the evidence either in the form of dots (Child 6), short vertical lines (Child 8) or, in the form of closely written numbers (Child 1) with the result that, for example, her estimated position for 12 was pinpointed at 19. However, using such a strategy did cause Child 1 to question her outcome on one occasion — the only instance within this phase of the study where a child appeared to question the outcome of their initial approach to estimating and then revise it. Child 1 wrote a sequence of numbers between 0 and 25 to estimate the position of 25. The final position of the 25 took her beyond the middle of the 0 to 100 line:

> There... It's gone far too.... It's a bit dodgy. Because twenty-five is not past fifty... I should've done dots. (Y4.1, estimating 25, identified 27)

Child 1 corrected the outcome of her accumulation approach to first estimate the position of the middle of the line as 50 and then to half the line again to give her estimate of the magnitude of 25. Two features seemed to appear from this effort. The first seemed to point to a perceptual corrector, 25 could not have a magnitude that is perceived to be greater than 50, and in recognition of this, there appears to be the application of a strategic approach where the number line is halved and halved again to give an estimate of the position of 25. There will be more to say about the former in Chapters 7 and 8.

The indications from the above also suggest that children could use a range of approaches to estimate position. In some instances, accumulation was simply replaced by a guess.

> I estimated... cause I guessed. (Y4.14. estimating 55, identified position 54)
>
> I might know where the fifty-five should be around... here... somewhere.
> (Y4.6, estimating 55, identified position of 43)

In others, there was evidence of an appreciation of the magnitude of a number relative to an end point before providing a reasonable guess.

> Ninety-five is near to a hundred. I was near to a hundred. I just thought that's ninety-five straight away. (Y4.5. estimating 95, identified position of 93)

Evidence of the use of a strategic approach to estimate the position of a number was very limited. We have seen how it was used by Child 1, but there was one other instance of its use and that by Child 5 when estimating 75.

> It's about.... Three quarters away from... three quarters away [from zero].
> (Y4.5, estimating 75, identified position of 69)

3.4 Conclusions from the Preliminary Studies

The purpose of the preliminary studies was to gain some indication of the informal influence that the number line may have on the individual's conceptions of number, what they thought a number line was and their ability to estimate magnitudes on a 0 to 100 line. The results from the opportunity samples provided guidance towards the construction of the method to be applied during the main study.

The evidence suggests that the number line did not figure extensively in the respondents' general conceptions of number. This outcome was independent of whether or not they had experienced the number line, but there were some differences noted between the conceptions of the children and those of the students. The latter did provide some evidence of use of the term. Conceptions such as 'row' or 'line' were used in a descriptive way to illustrate how numbers appeared, but perhaps the most remarkable outcome from the students was the use of descriptive language particularly as it was associated with specific numbers.

A further feature that emerged from the preliminary studies was that the articulation of conceptions of number and the number line were relatively strongly associated with different kinds of frame of reference identified by Pitta (1998) and Gray & Pitta (1999). Distinctions could be made between descriptive and relational frames and in particular those that may be identified as specific, episodic and generic.

All categories of respondents provided comments that related to specific examples of the items discussed:

> Three. Blue. Own handwriting. Alone. Clear. There is a three in empty space.
>
> (Undergraduate 4, talking about *number*)
>
> Like a ruler. They tell you what numbers to put on.
>
> (Graduate 2, talking about *number line*)
>
> The one, the two, the three, the four, ... the twenty-two...
>
> (Cypriot Child 3, talking about *number*)
>
> When it's two. (Child Y4.8, talking about *number*)

Evidence from amongst the children also an association with other ideas or through the articulation of comments that could be related to other ideas:

> I feel like counting... I... (Cypriot Child 4, talking about *numbers 1-100*)

All of the Cypriot children gave specific examples of one or more numbers when asked to consider what came to mind when they thought of number. In contrast the English children's responses included no response (1/14), specific examples of numbers (4/14), examples which associated numbers with where or how they were used (Episodes)

(7/14), a general response, for example 'numbers' (Y4.9) or, in two instances, wider ranging responses that introduced a broad variety of number facets that could provide the basis for diverging considerations.

Articulating their notion of number through associations or episodes the Year 4 children referred, for example, to 'Counting' (Y4.6; Y4.10; Y4.14; Y4.1), 'Adding and subtracting' (Y4.3). Whilst the broader responses, indicated a variety of possible uses of numbers (Y4.4).

Such differences in the quality of responses, in that there were examples that were 'specific', 'episodic', 'general' and 'generic', reflect responses for the word number across the spectrum of interviewees. Typical responses to the word number from the Cypriot children and undergraduates were specific. The graduate students tended to respond with general statements, there was no immediate additional explanation to clarify the response — "The word number" (Graduate 2) — but there was also isolated evidence of generic and through reference to symbolism, proceptually orientated responses (Undergraduates 6 and 2).

The qualitative differences in the variety of the responses from the English children when responding to the word number was reduced considerably when they attempted to explain what a number line was. 12 of the 14 children thought of the number line in a way that evoked some kind of action. Their perceptions were related to episodes that they had experienced "you can count" (Y4.7), "you just start counting" (Y4.6), "you can jump" (Y4.12) "you can put a number down" (Y4.13), "you have got to work it out" (Y4.5), "you do little hops" (Y4.6). Only 3 of the children provided alternatives to this form of explanation. We have seen that Child 10 gave a descriptive response that was associated with a specific number line, whilst Child 4 provide a more generic response that recognised the infinity associated with the number line:

> The line goes wherever it wants to go — it just goes into what it wants to go into. (Y4.4)

Given that responses to the notion of number line were dominated by reference to actions, it is hardly surprising that the children's dominant approach to estimation was an accumulation strategy — the application of an action (counting).

The tendency of the children to identify the visual presentation of the number line and the abstract notion of number through explanations associated with either specific or episodic terms suggests that within the main study the children selected for interview should be identified on their predisposition towards articulating their perception of items in a particular way. At issue is whether or not the children's perceptions are personal constructions or simply memory banks of episodes that they have experienced within their classrooms. Do children that have a particular way of articulating their perceptions of items possess different levels of understanding of the concept and use of the number line?

Given that there is such a commonality about the children's perception of the number line and given that the evidence suggests that these perceptions are strongly associated with what can be done, it would seem that the main study should also include a focus on the type of emphasis placed on this particular representation by the children's teachers. Therefore, it would also seem appropriate to consider the way that the number line is used within the classroom.

The interesting distribution of the estimates associated with identifying magnitudes suggest that this is worthy of further study. The distribution of the differences of the undergraduates and the graduate students suggest that with age the differences between estimation and actual magnitude are small. Also, it is interesting to note that they are all underestimates. The distribution of the errors associated with the two samples of children suggest some change because of age, but interestingly the distribution is similar whether or not the children have experience of establishing magnitudes on the number line. This suggests that an examination of the magnitude and distribution of differences throughout the primary school should be considered.

The outcomes of the exploratory study and the pilot study led to the formulation of the central question for this study:

> *How is the number line used within the English primary school and how is it understood by children?*

From the results of these preliminary studies, three main themes now guided the development of a response to this question:

- What are teachers' perceptions of the number line and how do they use it within the classroom?
- What do children construe from the lessons?
- What are children's perceptions of the number line and what sort of embodiment do they possess? To what extend do they recognise the underlining notions of a repeated unit and continuity and what is the pattern of the distribution of their estimates for magnitudes that are equidistant from the extremes.

Chapter 4 places a perspective on these issues and considers the development of the methods used within the main study.

Chapter 4: Methodology and Method

4.1 Introduction

The aim of this study is to attempt to gain some insight to the influence of a particular representation on young children's mathematical development. The emphasis of the study focuses on the number line and using the distinction noted by (Herbst, 1997) there is an attempt to distinguish between the teacher's presentation and use of the number line and children's use and understanding of it. It is a study of a key-representation within an innovatory programme, the National Numeracy Stategy (NNS) (DfEE, 1999a), that is in use in English schools. Though it focuses upon the use and understanding associated with one representational element of the innovation, the number line, the study will draw upon methods that are mainly associated with case studies but also will utilise some features of action research and enthnography to evaluate the relationship between use and understanding associated with its presentation.

In addressing the above issues, two related aspects become features of the research question:

(i) What indications are associated with teachers' presentation of the number line

(ii) Do the children's construals of the number line support an understanding of its structure and use.

Within Chapter 2 we saw that the review of the literature led to the identification of several issues associated with the use of the number line in an English classroom, whilst the issues that would become a focus for the main study were identified within §3.4.

This chapter reviews the method used to respond to these issues.

4.2 Theoretical Considerations

4.2.1 Background Consideration

The purpose of this study is to view things from a constructivist perspective, and therefore it is largely qualitative in nature. However, it recognises that the subjective construction of the child is associated with the "objective" presentation of the teacher. The distinction between subjective and objective is made from the differences manifest through the personal construal of an aspect of knowledge, and an indicative world, which transmits knowledge which the indicator expects to be known. Such a situation arises in the implementation of, for example the NNS which seems to portray an objective reality of mathematics — there is a body of mathematics that is potentially knowable to all — and the subjectivity that arises from the personal construction of this objectivity by the teacher and the child. Thus, the use of the term objective within the teacher context does not remove the recognition that the teacher himself/herself also makes a subjective construction of the issue to be considered. It is a conjecture of this study that what appears to be objective, in the sense that it is presenting a world that is knowable, in this instance the knowledge and use of the number line, is strongly influenced by subjective interpretation. Thus, the child receives from the teacher indications of an objective reality tempered by the teacher's subjectivity from which they in turn construe their own reality.

Thus within this study there are three aspects to the research. Essentially the study is a case study of an individual school and its implementation of a single representation designed to support a rise in children's achievement in mathematics. However, in a departure from the norms of case study research, the study also draws upon elements of action research to investigate the learning situation and evaluate the effects of the use of a representation portrayed by teachers and used and understood by children. The third aspect has an ethnographic flavour in that there will be an attempt to interpret the actions and construals of the children in terms of the indications of the teacher. Inevitably, the study will draw upon interviews and classroom observation to form its database.

4.2.2 Design Options

Through the analysis of the literature, the evidence suggests that a study of this form has several mechanisms through which its data may be obtained. Test, questionnaire, interview and observation, as well as a combination of two or more of these may produce data that enables a response to be made to the issue. Since the study draws upon three of these, questionnaires, interview and observation, it has a mixed approach in that it draws upon both qualitative and quantitative data.

Both approaches have their strengths and weaknesses. The general distinction between the two is that in qualitative studies the theory emerges from the data, whilst in quantitative studies it is generally generated before the data collection takes place (Lincoln & Denzin, 2000; Bryman, 2004). However, in this study the quantitative data is used to support the qualitative in an attempt to generate theory rather than to prove it.

Qualitative research involves an interpretive naturalistic approach to the world by emphasizing the qualities and meanings of entities studied in their settings, whereas quantitative studies emphasize measurement and quantification that support the use of descriptive and inferential statistics tests, but the settings can be artificial. Usually quantitative studies have a distinctive (structured) research instrument, for example a questionnaire or a test, which can provide numerical data in the form of categories or quantities for objective analysis. The structure within qualitative studies can seem less apparent, particularly if the instrument used is clinical interviewing since the quality of the data is usually derived in the form of words.

Qualitative research can be criticized as unscientific, since it may not have the scientific respectability of quantitative research. Statistical tests in quantitative research give researchers additional credibility and confidence in their findings and allow the possibility that others may confirm or refute them (Denscombe, 1998). Qualitative research can also be criticised as subjective, mainly because the researcher is the main instrument of data collection and analysis and the findings are based on his/her subjective interpretation. For this reason, qualitative research results are difficult, if not impossible to replicate and generalise, something always aimed for and desired in quantitative research (Cohen, Manion & Morrison, 2000; Bryman, 2004).

In qualitative research, the researcher wants to be close to their subjects in order to gain an insight into their way of thinking, which is the focus of this study, but it has been suggested that this relationship with the participants may influence the results (Lincoln & Denzin, 2000; Bryman, 2004). On the other hand however, the more detached approach of most quantitative research can fail to make people and social institutions distinct (Bryman, 2004).

A quantitative approach was used by Carr & Katterns (1984) to study children's understanding of the nature and the principles of the number line, whilst Merenluoto & Lehtinen (2002) used tests to investigate low and high achievers' level of understanding of the number line's density. Each reached their conclusion through a quantitative analysis of the outcome of the tests. Tests were also used by Merenluoto (2003) on the cognitive conflict created when the learner progresses from natural numbers to rational and real numbers on the number line.

Qualitative approaches have made use of structured and semi-structured interviews to investigate pupils' knowledge of mixed numbers and improper fractions and their ability to re-unitise number lines (Baturo & Cooper, 1999) and to investigate the compatibility of children's whole number and fraction knowledge with the use of the number line. Bills (2001) combined observation of teachers and interviews with children to consider the influence of classroom activities reflecting the use of a variety of representations on children's mental representation in the field of elementary arithmetic.

4.2.3 A Mixed Methodology

This study uses a mixed methodology. Investigating the use of the number line in school involved classroom observation, focusing on the teacher's use of the number line in classroom, and a follow up interview with children, investigating their personal knowledge and understanding of the number line as a result of instruction and use.

Gaining an insight into the child's perceptions and way of thinking about the number line required one-to-one semi-structured and semi-clinical interviews on a selected sample of children. Both the observation and the interviews were designed to provide qualitative data.

To provide extensive information about the understanding of a relatively large sample of children's perceptions of the number line, a questionnaire was designed. This led to quantitative data that was then supported by a quantitative analysis obtained from interviews with selected children. Thus, the overall research design developed and administered instruments that would provide both qualitative and quantitative data to address the research issues.

- To identifying teachers' perception of the number line and how it is used in their lessons, class teachers were interviewed and observed during their teaching. (Note: Teacher trainees were intended to be the initial group of teachers observed and they were given a questionnaire to identify their perceptions of the number line. However, difficulties associated with actually carrying out the observations meant that the study changed direction and became a case study that involved teachers and pupils within one school. See also §4.4 and §4.4.1)

- To obtaining a broad range of quantitative data and enable statistical test analysis, a range of pupils within different year groups, were given a questionnaire (§4.3.3.1) associated with interpreting the number line and its features.

- To establishing children's interpretation and understanding of the use of the number line in a particular lesson, semi-clinical interviews were used with a selected sample of children within each class (§4.3.2).

Because of its qualitative aspect, the study will emphasise the emerging characteristics that underscore interpretations of the number line by the respondents. The meanings that they have established will be given through words and any emerging theory will arise from the researcher's interpretation and classification of these words. The analysis of the results of the preliminary studies (Chapter 3) indicated that different people experience, perceive and conceptualise the number line phenomenon in qualitatively different ways and it also suggested the development of increasing sophistication in the varied qualities of this thinking. Since a central feature of the study is the classification and analysis of children's interpretation of the number line and its properties, this would suggest that the analysis of the qualitative data is oriented towards a phenomenographic perspective.

4.2.4 A Phenomenographic Orientation

The word phenomenography is derived from the Greek and it is an amalgam of two words: "grapho", which means to write/describe and "phenomenon", that which will be brought to light and made clear. It is therefore an approach that describes things which have been brought to light and made clear (Neuman, 1987, 1999).

Developed within a research group in the Department of Education at the University of Göteborg in Sweden in the early 1970's, phenomenography is described as

> ... an empirically based approach that aims to identify the qualitatively different ways in which different people experience, conceptualise, perceive, and understand various kinds of phenomena. Within this framework, learning assumes a central importance, because it represents a qualitative change from one conception concerning some particular aspect of reality to another. (Marton, 1988a; cited in Richardson, 1999, p. 53)

Phenomenography is a process of discovery; it can neither be called a theory nor a methodology (Neuman, 1987, 1999). It is a research specialisation that aims to address a particular kind of questions in an educational environment. Marton & Pang (1999) suggest that typical questions may be "What are the different ways of experiencing a phenomenon?" and "How do different ways of experiencing something evolve?" Notwithstanding its application in diverse studies such as management accounting (Rovio-Johansson, 1999) and the economic phenomena of price and trade (Pong, 1999) as well as science education (Reyes, 2001). Within the field of elementary arithmetic its approach has had a range of applications: considering the way in which young children handle arithmetic problems (Neuman, 1999), children's understanding of fractions and percentages (Runesson, 1999), the kinds of mental representation associated with young children's knowledge of elementary arithmetic (Pitta, 1998) and the relationship between classroom representation and children's handling of arithmetic (Bills, 2001).

The important objective of the phenomenographic approach is to understand and define the qualitatively different ways in which the phenomenon under investigation is experienced and conceptualised. What it is that is experienced and thought is viewed from the researcher's eye during an interview (Neuman, 1987; Runesson, 1999) whilst identification of the characteristics that arise from the analysis of interviews leads to "units of description" and "ways of functioning" (Marton, 1984, p. 19; cited in Neuman, 1999, p. 3). The units of description may be called conceptions and are viewed as the particular ways in which things appear to the individual. Neuman (1987) indicates that the more individuals are interviewed about a certain phenomenon, the more likely that most conceptions of that phenomenon will come to surface.

4.3 Research Instruments

4.3.1 Observations

In this study, observation would provide the link between teaching and learning, but it did prove to be the stumbling block for the development of the study (see §4.4.1) in terms of establishing the sample. Initially the sample of teachers and pupils was going to be defined through the teacher trainees and the schools they were assigned during their teaching practice, but the practicalities associated with this approach required a change in direction so that it involved one school and the teachers and children within that school.

The observations of the main study were focused on the teacher and the classroom as a whole, investigating the way the number line was used by teacher and pupil. The nature of the observations was simplistic and semi-structured: simplistic in the sense that they aimed to gain a sense of the way the number line was used, its context for this use, children's interpretation of the use and pupil-teacher interaction; semi-structured because of the wish to observe the application of suggestions by both the National

Curriculum (NC) and the NNS and the way these are put to practice. Thus the teacher's view of the benefits or otherwise of the use of the number line and their view on the importance of the model were of great interest.

4.3.2 Interviews

Interviews provided the data that could be considered in a qualitative way to support the phenomenological philosophy being used within the study. Interviews were not only used with children, but also with teachers (§4.4.4). In this section, we will focus on the interviews involved with the children.

A modified version of the "clinical interview" was used during the interviewing process. Ginsburg (1981) indicated that this is a flexible, unstructured and open-ended method of questioning and suggested that the approach was good for understanding and explaining children's mathematical knowledge, since it values the fact that different subjects may give different responses to the same question and therefore the clinical interview approach can treat them differently on a one-to-one basis. With this method, instructions can be clarified, the form and nature of the questions can be altered and there is the possibility to build on responses — something that cannot directly be achieved with a questionnaire.

The modification applied to the interviews within this study meets the requirements of being semi-clinical in that there is a combination of "structured" and "open interviewing" techniques (Cohen & Manion, 1985, p. 309). The structured component concerns the fixed "interview schedule" with questions, which were used at the start of each interview, such as "What did you do in numeracy today? Can you give me an example?" The questions used within this phase where directly associated with the issues that the study aims to address, for example, the relationship between the teacher's use of the number line and the children's understanding of the teaching.

However, the interviews took a different turn once a subject gave their own individual response. The interviewer would build on the interviewees' response and guide the course of an interview towards a broader conception of the children's understanding of the number line. This was the semi-structured component, during which questions were rephrased and their order amended (Cohen & Manion, 2000). During this phase of the interview, it was the comments from the children that guided the way within which the interview developed.

During the main study, a selected group of children were interviewed over a period of six months. This approach was considered to be the most appropriate for exploring and gaining a deeper insight into the individual's interpretation of the number line and the quality of their thinking associated with it.

The interviews used in the main study were in two forms: those associated with a set of questions (see Appendix IV) and those associated with the observed lessons. By associating interviews with the preset set of questions, the objective was to understand the way of thinking of the individual behind a question. The data received from the phase of the interview associated with the questionnaire proved to be particularly interesting and following consultation with teachers the questionnaire was given to all of the children within the observed classes.

In associating interviews with the observed lessons, the objective was to consider what children construed from their lesson and what their focus of attention was. Each interview had a common starting question or statement. For example:

> *Do you remember what you did in numeracy today? (and after their first response) Could you give me an example?*

In some instances, the children spoke of generalities associated with mathematics but did not focus upon the particular lesson. On these occasions, the interviewer prompted the children with questions or statements such as:

> *What else did you do in maths?*
>
> *What did you do when you were sitting on the carpet? You had some numbers...*
>
> *What did you do in the beginning of the lesson?*

You were doing addition... which ways of addition...

You were in the classroom and then your teacher did...

Your teacher had something on the board...

4.3.3 Questionnaire

In this study, there were two questionnaires. One was designed and given to trainee teachers (Appendix V), with the objective of gaining an insight into their understanding of the number line representation and also act as a selector of some teacher trainees who would be followed into school for observation (§4.4.1) and the other was given to the children (Appendix IV).

Reasons for the use of questionnaires were:

- To get an overview of the trainee teachers' understanding of the number line
- To get an overview of the child's understanding of the number line

The questionnaire developed for the teacher trainees followed the line of questioning that had been used with the undergraduates and the graduate students within the preliminary studies. Thus, it sought to examine what the words 'number' and 'number line' evoked and examined their perceptions of a range of number line segments and whether or not they could place additional numbers on presented number line segments. The outcome of the responses to these questionnaires was used to select a sample of trainees who would be observed. Although this aspect was not followed through, the results of the questionnaires are worth considering since the results do place a perspective upon teacher understanding.

The questionnaire given to the children focused upon specific aspects of the research questions and therefore considered three themes:

- Respondents' visual preference of the number line orientation

- Respondents' conception of the number line as a representation of the number system
- Respondents' skill in pinpointing numbers on the number line

4.3.3.1 Development and structure of questionnaire

The questionnaire went through several alterations before its final stage. Originally developed during the exploratory study and modified first for the pilot study and then once again after the pilot study.

Based on the NNS objectives (Appendix I) and NNS examples (Appendix II), the questionnaire (Appendix IV) was constructed. This was given out in two different ways: (i) as a 15-minute questionnaire for Years 2 and 3 (questions 5-9 only) and (ii) as a 25-30 minute questionnaire for Years 4, 5 and 6 (all questions). The objectives and origin of every question is explained below.

Question 1 (Figure 4.1) was influenced by Freudenthal (1973), who recommended that vertical number lines or even inclined number lines should be used with children, since their experience of number lines would be in a more traditional way, such as the thermometer which is vertical. This question was also influenced by Rodriguez, Parmar & Signer (2001) who, interested in the concept of movement along a line, gave a collection of horizontal, vertical and inclined orientations to culturally and linguistically diverse students. The purpose of this question was to gain an insight into the respondents' visual preference of a number line and investigate whether or not children had mental stereotypes or embodiments of a particular a number line orientation.

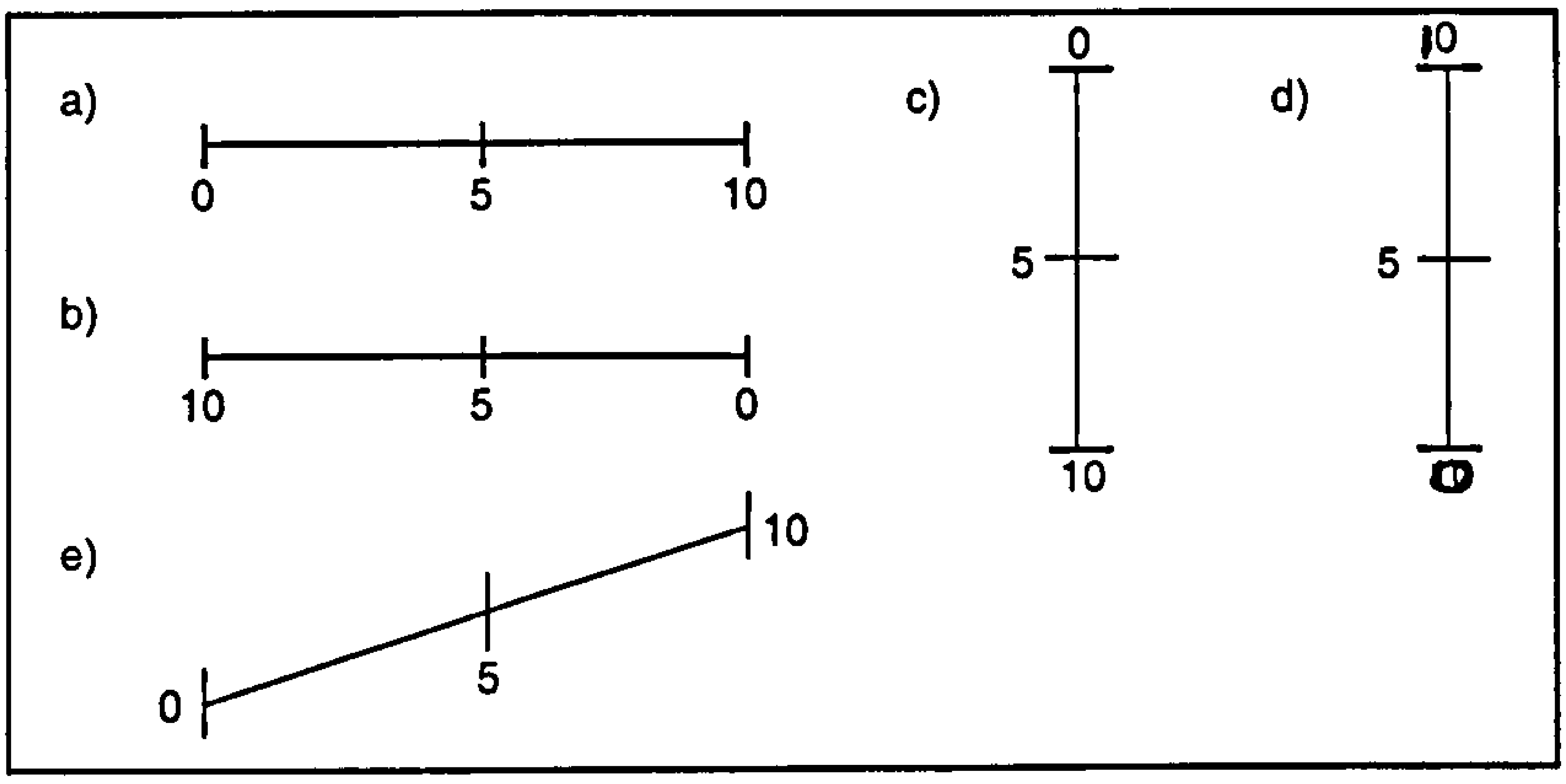

Figure 4.1: Tick the line you like the most. Explain why.

Question 2 (Figure 4.2) was inspired by the NNS (Section 3, pp. 14, 22, 26; Section 5, p. 23; Section 6, pp. 11, 15, 28) which indicated that children should be able to order and pinpoint fractions on the number line. Its purpose was to identify whether children appreciated that numbers could be placed within intervals that were identified.

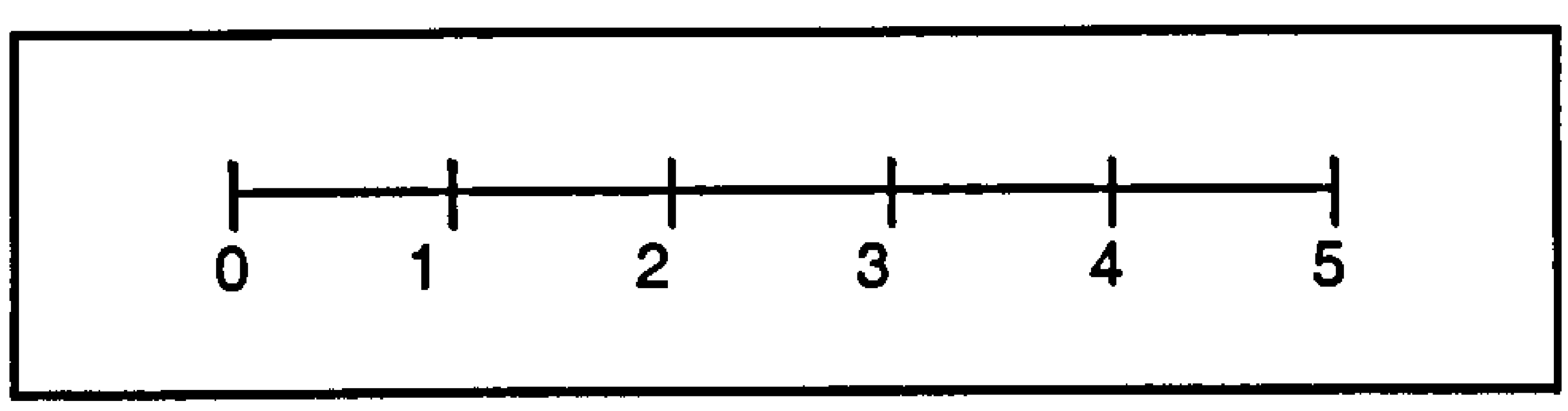

Figure 4.2: Can you put more numbers on the line?

Question 3 (Figure 4.3) was inspired by Beishuizen (1999), who gave to Dutch children empty number lines and asked them to either draw sums or make jumps from one number to another on the lines. These children tended to mark the numbers at points other than the ends of the line. Therefore, the question was included to see if the English children's embodiment of a number line segment included numbers pinpointed at either end or whether they placed the numbers at other points on the segment.

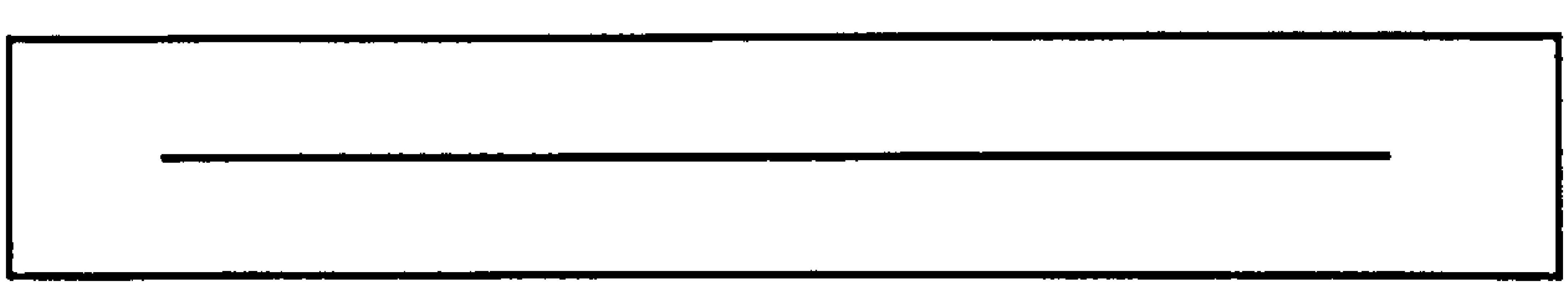

Figure 4.3: Put 0 and 20 on this line.

Question 4 (Figure 4.5) was inspired by NNS examples (Section 4, p. 12; Section 5, pp. 10, 11, 14; Section 6, pp. 14, 15, 28) where number tracks and marked number lines with missing numbers are presented. Its purpose was to identify the range of numbers the children used to complete the line (whole, decimals, fractions, negative, etc.).

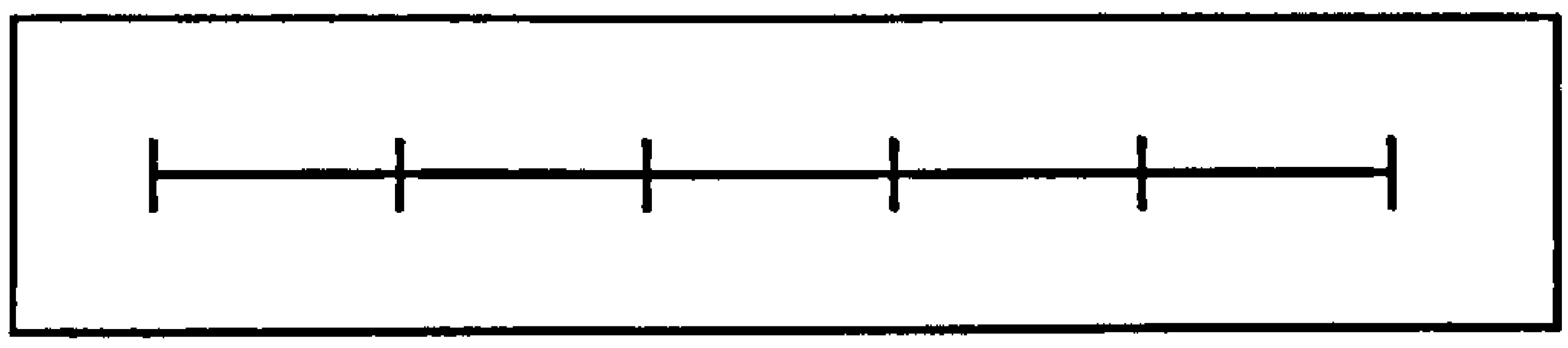

Figure 4.5: Put numbers on this line.

Questions 5, 6, 7, 8 and 9 originate from within the researcher's MSc dissertation (Doritou, 2001). The tasks involved either (i) pinpointing positions or (ii) recognising positions. For pinpointing position, descriptions of examples from within the NNS were followed (Section 3, pp. 6, 14; Section 4, p. 8; Section 5, pp. 8, 9, 17) and, using an unmarked line (Appendix IV, Q.9), children were invited to estimate the position of the numbers: 93, 45, 12, 5, 75, 3, 7, 25, 97, 95, 88 and 55. The series of numbers included pairs that were equidistant from either 0 or 100, for example 3 and 97. This was inspired by a comment within QCA (1999a):

> An important aspect of having an appreciation of number is to know when a number is close to 10 or a multiple of 10: to recognise, for example, that 47 is 3 away from 50, or that 96 is 4 away from 100. (p. 28)

The purpose of the question was to identify whether the accuracy of the positioning would change, as the numbers grew larger.

For recognising position, again descriptions of examples stated within the NNS were used (Section 5, p. 17; Section 6, p. 11). Designated lines, with the extremities marked, each had an arrow identifying a number position on each line (Appendix IV, Qs.5-8). The children were invited to estimate the number at which the arrow was pointing. The actual values represented by the arrows in four discrete questions were: 82, 12, 95, and 5. The purpose of the questions was to identify whether the children's accuracy at pinpointing on a segmented line with only the extremes as points of reference was different to the accuracy identified on a similar line but with the position of the number identified.

4.4 Sampling

Following the first phase of the exploratory study with doctoral students' (from several disciplines) understanding of the number line, the second phase of the exploratory study involved undergraduate students who were teacher trainees and children within Cyprus. These two samples were essential opportunity samples but the purpose was to investigate whether or not qualitative differences in understanding of the number line could be identified and whether or not questionnaires would supply relevant information. Subsequently, a modified version of the data collection process with children was trialled within an English school. The outcome of the analysis of these results supplied information that would prove to be influential in identifying the sample of children reported upon within this study.

There were two stages in the sampling process for the main section of this study. The first focussed on the use of final year trainee teachers within a large Midland University. However, after going a considerable way in identifying the sample and establishing a research relationship with individuals, because of practical difficulties the designed study had to be considerably modified. The original intention of using the trainee teachers was abandoned in favour of using one particular school. This section reports on both samples: (i) the trainee teachers and (ii) the teacher samples drawn from the school. The former is included because after going some way down the road towards obtaining information from the trainees, it is considered that the information obtained provides a sound background towards understanding the subject knowledge of general primary school teachers at the point of entry into the profession and complements the data obtained from the interviews with practicing school teachers at the school that agreed to participate within the study.

4.4.1 Teacher Trainees

It had been the initial intention of the study to relate the teaching with the number line and the children's interpretation from this teaching with observation of a selected sample of final year teacher trainees during their ten week final school experience and then, within selected schools trace a sample of children's development over the course of a further two terms. To meet this intention the full final year cohort of BA(Ed) students within the Education Department of a large Midland University were invited to participate in the initial phase of the study, which was an assessment of their understanding of the number line and its use. By this stage of their training the trainees were fully conversant with the contents of the National Numeracy Strategy, had experience teaching it within school and been provided core lectures associated with the number line.

To establish their initial understanding of the number line the students were invited to complete a questionnaire (Appendix V) developed from the experience within the exploratory studies.

From the analysis of the responses, a group of students were identified who were invited to participate further in the study by permitting the researcher to observe any teaching associated with the number line. After meeting with this group, ten agreed to further participation. For practical reasons an attempt was made to ensure that there may be more than one student in a school and that the schools themselves were not too divergent. Once the students were identified, all schools were visited and all but one agreed to take part in the study by allowing the researcher access to classes and children. The result was that nine students formed the core of "teachers" who would participate in the study.

Unfortunately, things did not progress as planned. Although the selected students had agreed to identify, from their forward planning, times when they would be making use of the number line and notify the researcher, for whatever reasons this process did not work in practice and even when it did, subsequent changes to a day's work could mean a fruitless journey by the researcher. Reasons for this were for example that although the student had planed a lesson on the number line and notified the researcher who then attended the school, on that particular day, school changes might replace a mathematics lesson with a school assembly. Consequently, the lesson would be cancelled or its context altered. Another reason was that some of the students never got back to the researcher either because they forgot to notify her or because they did not plan to use the number line in their lessons. In one occasion however, one of the students did notify the researcher, but instead of observing a lesson involving the number line, a form of number square was involved.

As we have discussed in the literature chapter, the number line has been identified as an important form of representation within the National Curriculum and the National Numeracy Strategy. A significant methodological feature of the study required free access to classrooms particularly when the number line is being used. It then requires free access to pupils. After approaching several schools, one school agreed to provide this access. Thus, the direction of the study changed to include observation of practicing teachers that taught mathematics using the National Numeracy Strategy as a key resource. Teachers within each year group within the school agreed to overall arrangements and indicated their willingness to notify the researcher whenever they used the number line as a key representation within their teaching. In one sense, this proved to be a fortuitous change. The school itself may be described as the type of school that the NNS may have been most beneficial for. Mathematical standards within the school did not meet the average although throughout the period of observations it became apparent that the teachers were very conscientious in attempting to raise these.

4.4.2 The School Background

The school is located in a deprived area of the West Midlands, with an average number of 270 pupils. The children come from different backgrounds with a quarter speaking English as a second language. More than half of the children move in and out of school yearly, two thirds of pupils (above the average) are entitled to free school meals, a third (above the average) have special educational needs (such as learning, behaviour, emotional, communication difficulties). A high number of pupils have been classified to be "at risk" on the child protection register, as far as their welfare and well-being are concerned. The school struggles to involve parents in the children's education, by informing them about homework, behaviour and attendance issues which need to be given their attention (OfSTED, 2004).

The 1999 OfSTED report gave satisfactory grades to the school in mathematics, science and English and unsatisfactory grades in information technology. In the results of the National Curriculum tests in English, mathematics and science the school's performance was graded with an E (Well below average) compared with similar schools which were graded with C (Average) or D (Below average). In contrast, the 2004 OfSTED provided an overall assessment of 'Good', signifying that teachers had good subject knowledge and that the teaching and the mathematical development have improved considerably. Despite this, the school's general standards are quite low, not only in mathematics but in general, with below national average achievement compared to other schools. The results of the National Curriculum tests at the end of Year 6 in English, mathematics and science have been below average for the previous three years (up to 2004), while those at the end of Year 2 in reading writing and mathematics are again lower than the national results.

Despite these results, within the 2004 inspection report it is stated that the curriculum has been implemented successfully and that it meets pupils' needs satisfactorily, enabling all groups of pupils to achieve well:

> The National Literacy and the Numeracy Strategies have been fully and successfully implemented. These have been the main foci of teaching over the past four years.
> (OfSTED report 2004, p. 13)

and that it is time to shift attention away from literacy and numeracy because:

> In recent years there has been a high focus on literacy and numeracy and the school has now identified the need to improve the provision for, and the links between, the non-core subjects. (OfSTED report 2004, p. 7)

A main reason for choosing a school with below average achievement was not only the results of the preliminary studies (which involved children of average and low achievement) but also the fact that it would provide some indications of the way that the NNS, which is designed to raise standards, is developing shifts in thinking amongst the more disadvantaged groups.

4.4.3 The Interviewees

Classroom lessons were observed and a short follow-up interview, based upon the use of the number line at that time was conducted with the selected children.

The purpose in establishing the samples was to:

(a) Take snapshots from within different age groups to obtain some sense of the development and use of the number line and

(b) Investigate the relationship between children's understanding and the use of the number line.

To respond to these issues two distinct samples, each having different purpose, were identified:

1. Children within Years 2 and 3 who are experiencing the introduction and use of the number line in the development of whole number arithmetic. Since the emphasis during these two years was the use of the number line as a tool, through these interviews it was the intention to examine the children's procedural understanding in the context of elementary arithmetic and whether or not this understanding gave some insight into the conceptual aspects of the number line, particularly whole numbers.

2. Children within Years 4, 5, 6 who have had extensive use of the number line and its associations with whole number, decimals, fractions and negative numbers. It is this sample that may provide a sense of the relationship between understanding and use of the number line and the children's developing understanding of the number system.

The method of selecting the children, who would be invited to participate within the main study interviews, grew out of the discourse of respondents within the pilot study phases and the way the respondents talked about their perceptions of the number line. It became clear that there was a distinction that could be identified in the context of the descriptive and relational frames of reference that were associated with mental representations (Pitta & Gray, 2000). They described descriptive ones as those with 'episodic' and 'specific' characteristics whilst relational ones projected 'generic' and 'proceptual' characteristics and suggested that a tendency towards one form allowed individuals to do things that the other does not:

> 'Episodic' and 'specific' mental representations support the communication of descriptive elements. In global terms they provide something to talk about. In arithmetical terms it is conjectured that they generate things to do. The more relational mental representations provide the basis for recognizing intrinsic qualities through which we may form connections between items. In arithmetical terms it is conjectured that these support the development of transformations that build upon these qualities to provide alternatives which support success. (Pitta & Gray, 2000; pp. 314-315)

The descriptive/relational characteristics associated with discourse of the number line by respondents within the exploratory and pilot studies suggested that one method of selection of the children for the main component of this study could be on the basis of an individual discourse. Consequently, it was decided that a shortened version of Pitta's (1998) word item bank should be trialled and used as the prime source of selection of the children. It was also conjectured that selecting children on the basis of the articulation associated with the items would serve the added purpose of identifying children who felt comfortable communicating what they thought.

A reappraisal of Pitta's (1998) data indicated that four words, two common nouns, '*table*' and '*ball*', and two abstract nouns, '*five*' and '*number*' proved to be the best discriminators in identifying descriptive and relational differences. The evidence suggested that whilst all of Pitta's respondents initially articulated descriptive aspects of the items, in almost equal measure between episodic and specific characteristics, given an opportunity to extend their thoughts the children fell into two groups, those who remained descriptive and those who introduced relational characteristics into the discussion. The former tended to be the low achievers whilst the latter tended to be high achievers.

Following Pitta's approach, each child within the classes Year 1 to Year 6 was invited to indicate (this is one example of the four words):

What comes to mind when you think of the word number?

Responses were received in two parts, an immediate response and then children were given the opportunity to expand on this response. On the basis of the analysis of the responses, it was decided that more detailed interviews with Year 1 children were inappropriate since there were considerable difficulties in obtaining useful data although lessons associated with the use of the number line continued to be observed. Additionally two other factors were considered before the final selection of five children from each of the classes of Years 2, 3, 4, 5 and 6.

Although selection was mainly based on the quality of the responses, the verbal and written skills of children, especially for those within Years 2 and 3 played a significant role. Each class teacher was given the opportunity to consider the draft list of selected children. Perhaps one of the most interesting features associated with this draft list was the comment from the senior teacher within the junior section of the school (Years 3 to 6) that:

> Yes, that's quite right. You definitely have a range of all abilities there. If you would split them in three rather than five groups you would have high, average and low as you have estimated them. You haven't included any from the special needs though.

Test results from Standard Attainment Tasks (SATs), Qualifications and Curriculum Authority (QCA) and the teacher's personal assessment were also considered so that a sample of children representing those considered to be low, average and high achievers in mathematics was identified.

An example of the categorisation of children's responses, from asking Pitta's (1998) question, together with the researcher's conclusion, the teacher's opinion and an official test result (i.e. QCA or SAT) is in Table 4.1. This example is for Year 4 and identifies the four children participating in the study from this year group.

Word/child	Child 1	Child 2	Child 3	Child 4
Table	Episodic	Specific	Specific	Episodic
Ball	Episodic	Specific	Episodic	Episodic
Number	Specific	Specific	Episodic	Episodic
Five	General	Episodic	General	Episodic
Researcher's assessment	Above average	Average to Low	Average	Low
Teacher's assessment	High	Below	Average	Below
Official grade	3b[6]	2a	2a	2b

Table 4.1: Identifying children who would participate within interviews.

The researcher's assessment of the children was made on relative comparisons in the differences in their responses. Episodic responses were regarded as lower quality than specific responses, since they were associated only with an experience. Specific responses illustrated responses that could refer in relative detail to one aspect of the item under discussion whilst general responses were regarded as pivotal in that the respondent could expand on the item in either a descriptive or a relational way.

Using the approach identified above, four children from each of the classes Y2, Y3 and Y4 and five children from Y5 and Y6, giving a total of 22 children, were selected for the one-to-one interviews. This form of selection also had the advantage that any comments made by the children during interviews may be assessed against the general framework from which they were identified.

Generally, the pattern of the responses was used to indicate the most usual profile of a child. For example, a child who clearly gave responses of a similar type to each item was identified as a child who reflected that type so, for example, a child who gave four episodic responses was identified as episodic. Similarly, a child who gave three similar qualitative responses was identified in a way that reflected the dominant response. There was a tendency for some children to give episodic and specific responses so these were identified as specific/episodic. The problematic areas for consideration were

[6] Increasing-order (left to right) of achievement: W, 1c, 1b, 1a, 2c, 2b, 2a, 3c, 3b, 3a, 4c, 4b, 4a, 5c, 5b, 5a

where children gave three or even four (this happened in only one instance) qualitatively different responses. In those instances, the decision was made to let the most sophisticated response identified through the abstract nouns, *number* and *five*, indicate the final category of the child. In all instances this only happened when a mathematical response was identified as generic, examples being:

> Numbers are things you do sums with and maths and times tables. When you write a sentence like: I bounced the ball twenty-five times. You use them in sentences as well.
>
> (Child 2, Y3 - talking about *number*)

> Double it; ten. Times it; twenty-five. Divide it; two and a half. And you can't take it away because it equals a fraction. If you take-away five it equals zero, so you can take-away five. (Child 2, Y5 - talking about *five*)

> Five is a number between zero and ten. And it's between zero and one hundred. And you say fifty and it's got the ten that is a five. And it's a half.
>
> (Child 1, Y6 - talking about *five*)

> Half ten, it's in the five times table, if you say five times five it's a square number, fifty per cent of ten. (Child 4, Y6 - talking about *five*)

Year.Child	Frame of reference	Achievement
2.1	Episodic	2
2 2	Specific/Episodic	2
2 3	Specific/Episodic	3
2.4	Specific	1
3.1	Specific	2
3 2	Generic	1
3 3	Episodic	3
3.4	Specific/Episodic	2
4 1	Specific/Episodic	1
4 2	Specific	2
4.3	Specific/Episodic	2
4.4	Episodic	3
5 1	Specific/Episodic	3
5 2	Generic	1
5 3	Specific	3
5 4	Specific/Episodic	1
5.5	Episodic	2
6 1	Generic	1
6 2	Specific/Episodic	1
6.3	Specific	3
6.4	Generic	2
6.5	Episodic	2

Table 4.2: Frames of reference and achievement band of children selected for interview.

The responses of every child in the full sample were considered for the final selection. After the children who were to take part within the interviews were selected, their general mathematical achievement was considered through consultation with the class teachers and by considering their achievement in three broad bands identified from their attainment within the SATs, as mentioned earlier within this section.

Table 4.2 summarises the general characteristic of each child selected for interview. Each child, identified by a year group and a number within that year group, thus Child 3 in Year 4 is identified as 4.3, is associated with an overall category identifying their frame of reference and their general achievement within mathematics. What is particularly noticeable from the way that the children were selected is that 4 children (3.2; 5.2; 6.1 and 6.4) display a generic frame of reference. Child 3.2 is the only child below Y5 to be identified through a generic response because although his other responses were episodic and specific his response to 'number' was generic. Interestingly, he was identified as a high achiever. In general, the children identified as generic were either identified as high achievers in mathematics (5.2; 6.1) or average (6.4). However, Child 6.1 identified as generic was the only child in the full sample to give two generic responses to the mathematical items. Interestingly, responses to the concrete nouns were episodic. However, it was less easy to discern a relationship between children, where responses were largely episodic or specific, and their mathematical achievement. One child from each of Years 4 and 5 who gave these responses, was identified as a high achiever.

4.4.4 The Teachers

The teachers involved with the classroom observations and whose pupils participated in the study were interviewed. The final teacher sample contained teachers who taught mathematics within each of the year groups 1 to 6.

Structured interviews on a one-to-one basis were used to gain an insight into teachers' perspective of the number line. In particular, information on the following was obtained:

How important is the number line to your teaching in mathematics?

When do you use it and how? (Establish the main use)

Do you think it is a good representation of the number system?

Do the children have any difficulties with it? If so what?

4.5 Data Analysis

The collected data consisted of questionnaire, interviews and observation. Each aspect of the data was analysed separately and later common themes emerging from two or all three of these parts were brought together, especially in the chapter (6) on teacher development and children's perceptions from particular lessons. The transcripts from the children's interview data was viewed and analysed from a phenomenological perspective. Statements accompanied by incorrect answers are of particular interest.

In order to understand and categorize the different conceptions, that is the ways of thinking about the phenomenon under investigation (the number line), some use was made of categories identified by Pitta (1998) to make sense of children's construction of a lesson. The analysis of the questionnaire material is largely based upon descriptive statistics, but this is supported by an analysis of each question where categories of responses are emerging from the data itself. In the questions where children were asked to put more numbers on the lines, the categories were easily formed, since children's responses fell under the different kinds of numbers of the number system, therefore the categories identified were: (i) natural number (ii) rational number or equivalent (i.e. decimals) (iii) real number understanding of the number system.

In the case of last items associated with estimation, the items were analysed both qualitatively and quantitatively. Qualitatively in the sense that they provide quotes from children's explanation of their thinking when estimating. Quantitatively since diagrams and statistics are applied to the values the children pinpointed and to their errors from the position of the actual number. Mean errors associated with particular year groups were devised from the notion of absolute error to minimise the convergence to zero that may have occurred in some instances. This feature is more fully discussed within §7.4.3.1.

Similarly to the way in which the children's verbal indications of their understanding of the number line and the meaning that they extracted from lessons was considered, the data obtained from observation was categorised dependant upon the emphasis placed

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> We consider the category of description a discovery, why should we require two researchers to make the same discovery independently? On the other hand, once the discovery has been made we should certainly be able to communicate it, and other researchers should be able to use the intellectual tools that are supposed to be the outcome of this kind of research and be able to replicate and confirm our discoveries. Consequently, what we want to ascertain is that once categories of description are made explicit, other researchers should be able to identify them when they are applicable in varying contexts. In accordance with this, indicators of reliability should not be concerned with the extent to which categories are discovered independently, but the extent to which they are identified once they have been specified. (p. 251)

Thus, there may not be any reliability of discovery, only a reliability of identification. In actual fact, it is not the categories of description that are discovered but conceptions, identified as the result of the analysis of the data from each research instrument. That responses are reliable is established firstly through a sequence of interviews, secondly by the somewhat lengthy period of time between first and final interviews and thirdly through the range of questionnaire items used to elicit broad categories of response. Only after repeated analysis of the verbal comments, were the most powerful descriptive concepts and categories for discussion of the results identified.

Of course, not all conceptions will be "discovered". For example, the analysis of the pilot study has indicated the remarkable similarity between the quality of responses within that aspect of the study and those that were identified by Pitta (1998). As a result of such consistency, her categories have been used to identify one aspect of the results.

The issue of validity in a study of this sort is not any easier to resolve than that of reliability. The only way in which the interpretations themselves may be seen as valid or not is in whether or not they "make sense". If some of the notions cannot be understood as parts of a structured whole, the final model is not intelligible. If data collected by another researcher can be interpreted in terms of outcomes established, this makes both of the compared studies intelligible.

However, there exists another form of validation: validation of the responses given by the subjects. How do we know that the response given is a manifestation of the conception held by the subject and not simply a "random" answer or an explanation given to communicate in that moment? In order to clarify these questions, we can refer to Piaget's (1979) way of identifying "genuine" convictions:

- The consistency with which one child provides responses to the stimuli and the consistency among answers given by many children.
- The appearance of some form of evolution in responses.

In this study, the concern will not be to focus on providing an explanation or a cause for any one response. Rather, it has been to identify what a particular response means. The objective is to respond to all responses and to collectively draw these into the phenomena which may provide some account of the way children are given, receive and conceptualise the number line then to consider the consequences of this on their numerical development.

4.8 Limitations of the Study

This study aimed to investigate children's and teachers' use and understanding of the number line as a representation of the number system. An issue in this study has been the individual's, particularly the children's, private thinking which may not often be externalised. Despite the fact that the interviewer may ask children to put their thoughts into words and justify and reflect on their actions, there are cases where children have no direct access to their cognitive processes or a limited control over language and therefore fail to provide representative information of their thinking. For this reason, children were given a blank piece of paper, where they were allowed to write and draw, as a means of overcoming such issues and their conceptions were also considered through a variety of instruments.

Defining the sample has been the stumbling block for this study. The initial idea to involve teacher trainees delayed the research considerably. Nevertheless, it provided data that could be used as part of the teacher's conception of the number line. This proved fruitful in the end, since the teachers had just enough time for the researcher to arrange the classroom observations, but limited or no time to respond to a questionnaire or an interview. For this reason, only a few of the practicing teachers took part in the short interviews.

What was pressurizing at times for the researcher was having to ask, almost on a daily basis, the teachers whose lessons were observed whether or not they would be using the number line on that day. The use of supply teachers rather extensively within Year 6 did not help this aspect of the research. Whilst the full-time teacher within the class was aware, and fully supportive, of the researcher's interest on the number line and would inform her of relevant lessons, transient teachers would not. As a result, not enough observations were made for negative numbers represented on the number line.

At times it was revealed that teachers would either forget or would consider it unimportant to let the researcher know about such use, since it may only be a brief part (soothe introductory phase) of the lesson. In other occasions, the use of the number line by the teacher would be purely spontaneous during a lesson, consequently the researcher again would not have been notified. Sometimes the teachers would plan or modify a lesson to fit the needs of the researcher and would therefore include use of the number line. At other times, two teachers may have taught the number line during similar time slots. Thus, the researcher had to choose to attend one, according to the data collection needed at the time. A full reappraisal and reflection on the methods used is included with Chapter 8 (§8.4).

Within the next three chapters, the results of the study are presented. These are considered under three main themes; teachers' understanding of the number line, teaching and learning with the number line and children's conceptions of the number line.

Chapter 5: Teachers' Understanding of the Number Line

5.1 Introduction

One of the research questions identified for this study is associated with the teachers' perception of the number line and the way they use it within the classroom. This raises issues on teachers' beliefs, knowledge and understanding of the number line representation as well as mathematics activities with which they associate its use in their lessons.

In this chapter, teacher trainees' (also referred to as students) and practicing classroom teachers' perceptions and understanding of the number line are examined. The information received by the trainee teachers in their final year of training is considered, a reflection of that indicated by the practicing teachers, therefore data from trainee teachers and practicing teachers will be used interchangeably (see §4.4.1), although it is acknowledged that teacher trainees have less classroom experience than full-time teachers. This difference may be reflected within the conceptions each group has of the number line.

The trainees participating in this study followed a mixture of subjects, such as English, art, music and a third of them followed mathematics and science.

The key issues considered within the chapter are:

Teacher trainees' perception and understanding of the number line (§5.2).

Practicing teachers' perception of the number line as a representation of the number system and their indication of the way they use it (§5.3).

5.2 Teacher Trainees' Conceptions of the Number Line

The experience from within the pilot study had suggested that the Cypriot teacher trainees' conceptions of a number line were generally expressed in descriptive terms associated with their embodiment of a particular number line (§3.2.2).

The focus upon the teacher trainees' perception and understanding of the number line is derived from two issues:

- The first drew from the experience of the pilot study and attempted to establish the meaning that they gave to the words *number line* (§5.2.1).
- The second attempted to establish whether or not they could identify the underlying continuity that was associated with the number line (§5.2.2).

5.2.1 Defining the Number Line

69 Teacher Trainees (TT) in their fourth year of training and who were about to embark on their final experience within school, were asked as part of a questionnaire to define a number line (Appendix V, Q. 1).

Only one student provided a definition that implied that the number line was infinite and contained all numbers:

> A line that contains all rationals and irrational numbers. It is an infinite line. (TT4)

One other suggested it was:

> A continuous line of all of the numbers within our number system. (TT1)

Two others provided definitions that evoked either the notion of infinity but with no further explanation, two indicated that the number line was limited to rational numbers, whilst one other defined a number line with a response that may be interpreted as an association with magnitude:

> A sequence of numbers arranged on a line which has an infinite number of divisions.
> (TT23)

> A line of numbers on which any number can be placed. (TT48)
>
> A line where you may place all the rationals at some point on the line. (TT32)
>
> Representation of value according to how far the number is along the line. (TT43)

None of the above students gave any explicit reference to the notion of a repeated unit which could be partitioned, although partitioning may be implied from the statement of TT4. However, almost one quarter of the students (16/69) did make reference to some form of equal spacing associated with the line, although there was some evidence of little formality about the way they articulated this underlying feature:

> A line which is separated equally into different portions. (TT2)
>
> A straight line with equal distances marked. (TT7)
>
> A piece of apparatus with equal divisions marked. (TT10)

13 of these sixteen students associated the notions of equal spacing with numbers although in two instances the students referred to digits:

> A line with digits equally spaced along it. (TT47)
>
> A line with numbers attached at equal intervals. (TT66)
>
> A line which numbers are spaced evenly across it in a specified pattern. (TT17)
>
> An equally segmented line, each segment numbered in ascending order. (TT20)

Although it is not certain, TT20's definition suggests that she is thinking about a number line that only has positive numbers. This type of definition was relatively common:

> Numbers placed at identical intervals marked on a line in ascending order. (TT15)

and indeed, no student made explicit reference to the notion that a number line could contain negative numbers.

TT17's reference to pattern was, together with notions of order and sequence, a feature of the number line identified by 42% of the respondents:

> A string of numbers in a pattern. (TT27)
>
> Numbers in a correct order. (TT9)

> A sequence of numbers in a row. (TT22)
>
> A sequence of numbers ordered from left to right. (TT24)
>
> A line in which there is a number sequence reaching from lowest to highest number. (TT11)
>
> An ordered set of numbers in sequence, horizontal. (TT6)

Here again we see no explicit reference to negative numbers. The implications in two of these quotes (TT24 and TT11) suggest that the number line only contains whole numbers, an issue confirmed by the comments of some trainees:

> A line with number patterns on it — or from zero to a number. (TT12)
>
> Numbers that have been arranged in some form of sequence mainly from 0 to 10. (TT35)
>
> A horizontal line with a series of digits on it that have a pattern: one to ten; ten to one hundred. (TT42)

The above comments also give the sense that the number line is finite and although none of these particular trainees made any reference to the notion of partitioning the intervals, one student did:

> A horizontal line divided into ten equal sections allowing it to be divided into fractions or quantities. (TT64)

Interestingly, in addition to these students who explicitly mentioned order, pattern or sequence, six others introduced the word "chronological" to define the number line:

> A chronological line of numbers. (TT37)
>
> A line with marked number intervals in chronological order. (TT56)
>
> A horizontal line where positive numbers ascend in some sort of chronological order. (TT61)

We can see from the definitions provided by the trainees identified through the above examples, that reference to the underlying qualities of Herbst's (1997) definition — the consecutive translation of a segment U as a unit from zero, the partitioning of U in an infinite number of ways — is extremely limited. We saw that only three students referred to infinity, but only one of those implied that through partitioning all numbers could be represented. However, though there was no reference to the notion of "consecutive partition", as we have seen, almost 25% of the teacher trainees indicated that a number line possessed equal divisions. Their definitions appear to be founded upon partitioning rather than the continued replication of a defined unit.

Herbst further indicated that a number line can be formed by choosing a unit, repeating it from zero and then attaching to the end of each repeated unit a natural number. Though just over 80% of the teacher trainees associated the notion of the number line with a number or numbers, the majority of the remainder focussed on defining the number line as a tool (see below) but, as TT6 (above) indicated, there was also some evidence that the reference to numbers was not linked to the notion of line.

The overall impression left from this part of the analysis of the trainees' definitions of the number line was that they did not define it, but instead indicated how it may be seen. The sense was that they were describing a specific number line but often this specificity was limited to the more obvious perceptual characteristics rather than conceptual aspects of the line. In doing this, essential features were often omitted. Only in the first six instances quoted above do we see the trainees' explanations rise above specificity to give more sophisticated responses. In the case of TT4 — "A line that contains all rational and irrational numbers. It is an infinite line"— the quality of the response is "generic" (see §3.2.1). Several related concepts are introduced.

An additional feature of the trainees' definition of the number line was its identification as a tool. Almost 10% of the trainees indicated that the number line was used for calculation or to solve mathematical problems. Thus, we see the number line defined as:

> A continuous line in which numbers can be placed and used to aid calculations. (TT3)
>
> A piece of apparatus with equal divisions which children use to help them count. (TT10)
>
> A line with numbers on representing intervals, aid to solving mathematical problems. (TT34)

or associated with the notions of counting:

> A device to aid learning, involving counting on and counting back. (TT39)

A method used to count on or back horizontally. (TT62)

To aid children when counting up or down. (TT65)

In one instance, the identified process was left open to interpretation:

A way of roughly finding out any numbers between any two given extremes at each end. (TT52)

Although the above responses emphasise the nature of the number line as a "helping tool" (§2.3.2), and although Herbst (1997) suggested that its dense nature meets such a requirement, there seems little indication from these particular responses that the trainees could use the number line with flexibility and understanding. Additionally, the quality of the responses suggests that those students who emphasise use are drawing upon experience, either as learner or as teacher, and from this perspective, they were drawing upon episodes within that experience.

5.2.2 Student Teachers' Conceptions of Continuity

Within §2.4.1 we saw how Bourbaki (1984) indicated that the Greek conception of magnitude implied that numbers may be measured off on a line. From this perspective, the notion of continuity underscores the properties of a number line and that each point on the line could be seen to correspond to a unique number. Bright, Behr, Post & Wachsmuth (1988) suggested that activities associated with the creation of the number line; identification of a length representing the unit, iteration of the unit, subdivision of the unit, could reinforce the notion of continuity but that to make sense of any point on the line two other reference points need to be included.

To establish whether or not the teacher trainees recognised this sense of continuity, they were presented a number line as shown in Figure 5.1 and asked whether they could put more numbers on the line (Appendix. V, Q.4).

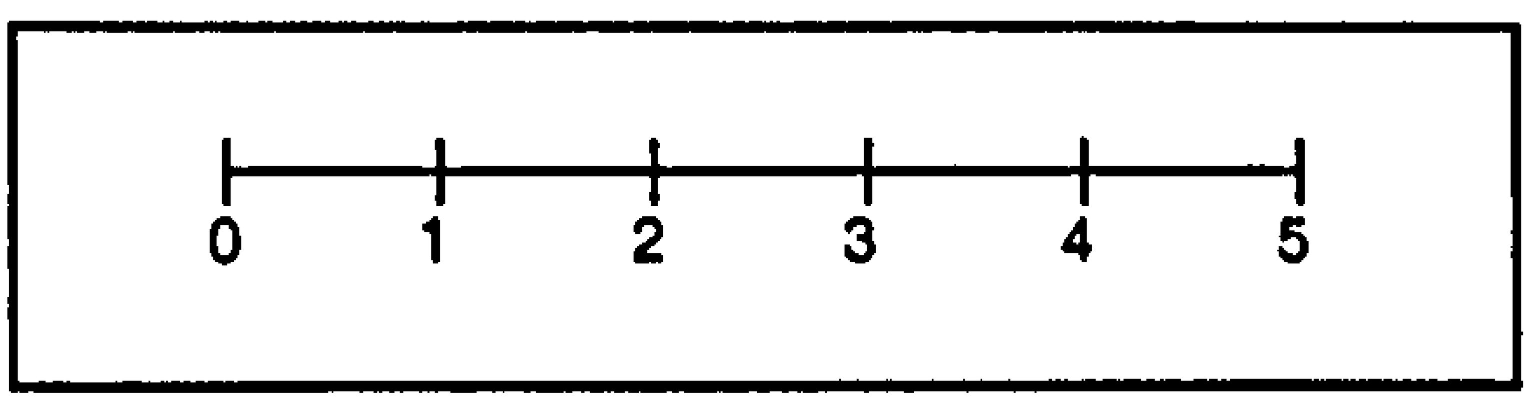


Figure 5.1: Item requiring Teacher Trainees to add further numbers.

All of the teacher trainees indicated that further numbers could be applied to the line in Figure 5.1. Whilst 80% suggested that addition of decimals and/or fractions, the other 20% indicated that an infinite number of numbers could be placed on it. This appears to be something of a contradiction with their earlier definitions of the number line and their responses to the following questions when asked to consider the item represented by Figure 5.2 (also in Appendix V, Q.2):

a) What do you think when you look at these lines? Write down your thoughts.

b) Can you put these lines in one line? Explain what you thought.

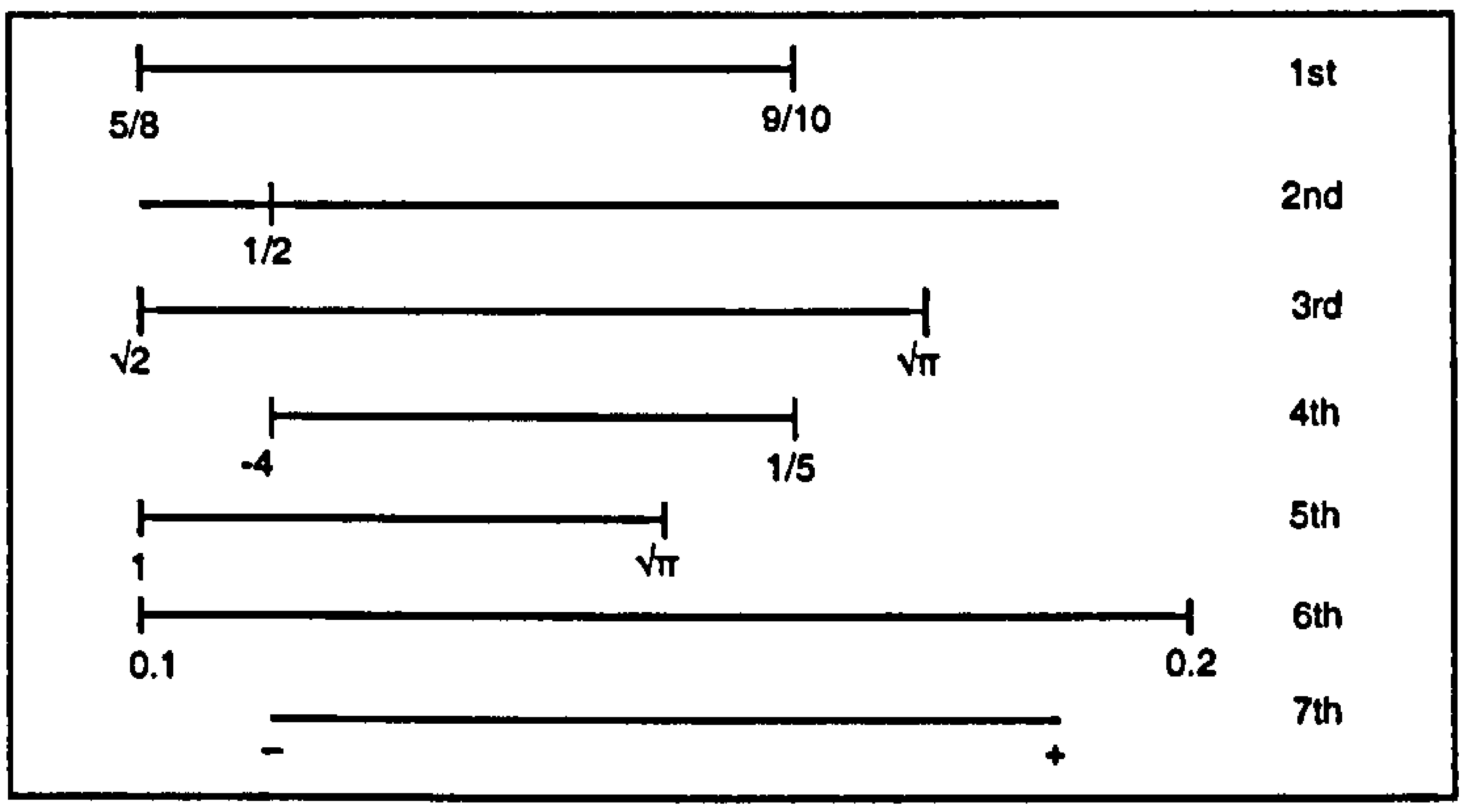


Figure 5.2: Item from Teacher Trainee Questionnaire

The number lines within Figure 5.2 had either little meaning for the teacher trainees or evoked descriptions of the numbers within which individual number line segments were framed. There were instances when none of the lines appeared to make sense.

> They don't mean anything to me. (TT9)

whilst in others, the trainees would identify those that made sense and those that did not. For example, TT61 indicated that Lines 1, 3, 4, and 5 made no sense to her but added:

> [Line 2] Makes sense, but some more numbers need to be written on.
>
> [Line 6] This one makes sense, but the scale between is too large.
>
> [Line 7] Positive and negative numbers.

The large scale associated with Line 6 was something that also concerned TT15, who also indicated that Lines 1, 3 and 5 didn't make sense to her, and suggested that Line 6 was a:

> Large interval for a small difference (TT15)

She was also concerned that:

> [Line 1 showed] a difficult use of fraction... [Line 2 has] no ending number so the interval is open... [within Line 3] the digits are not represented in the same format... [and since Line 7] had no digits it is therefore not useful. (TT15)

There is an evident hint of confusion arising in TT15's response to Line 1, since it included mixed fractions, a feature that caused confusion for other trainees:

> Confusing - starts in eights then changes to tenths. (TT13)
>
> Can different numbers like this go on the same line? Surely it needs to be all tenths or eighths. (TT57)

Line 6, even though it evoked comments that it was badly scaled, was more generally recognised as:

> More my type of number line! I understand this. (TT57)

Frequently, the lines were simply described and therefore appeared to make sense in terms of the number frames or the specific numbers represented upon them:

> Fractions [Line 1]. Fractions [Line 2]. Square root [Line 3]. Decimals [Line 6]. Negative/positive [Line 7]. (TT14)
>
> Fractions (maybe representing test results?) although not out of the same number [Line 1]. Un-balanced line? [Line 2]. Pi [Line 3]. Pi [Line 5]. Decimal line [Line 6]. Positive and negative [Line 7]. (TT20)

However, amongst a small minority of the trainees (approximately 5%), the marks on the lines provided the basis for comments that suggested a conceptual understanding of particular lines:

> Any number greater than 1/2 could be put here [Line 2]. Irrational numbers [Line 3]. Zero could be closer to 1/5 [Line 4]. Covers a small scale of numbers [Line 5].Covers all numbers [Line 7]. (TT48)

> Infinite line [Line 2]. Irrational numbers [Line 3]. Small intervals [Line 5]. Decimal number line [Line 6]. Line representing all numbers [Line 7]. (TT23)

Approximately half of the trainees were unable to give a response to the second item that invited them to consider whether or not all of the lines within Figure 5.2 could be placed on one line (Appendix V, Q.3):

> Not sure what this means. (TT49)
>
> I don't know where to start! (TT67)
>
> Sorry - I would struggle to do this! (TT68)
>
> No - I can't - too difficult! (TT61)
>
> No. I thought that number lines need to have similar numbers placed on them i.e. all fractions with the same denominator, integers, etc. (TT57)

The trainees who intimated that the lines could be merged fell into three groups:

- Those (23% of the full sample) that simply agreed that the lines could be merged, but did not attempt to do it:

> All numbers can be put on one line. (TT23)
>
> You would need to convert them all into the same sort of number, e.g. decimals. (TT36)
>
> Yes, you could apart from not the irrational numbers as you can't place them exactly. (TT21)

- Those (20% of the full sample) that focussed on how it could be done making reference to distance, order or continuity, but did not attempt to do it:

> Look for smallest and largest numbers first, then put the rest of the numbers in order in between at the right distance from previous and next number. (TT26)
>
> Yes. Take the lowest value, turn into a decimal and place on the left, take the highest value turn into a decimal and place on the right. Make sure they are placed in the correct positions on the line. (TT44)

> Yes, you only ever look at small sections of a number line, it is a continuous line which all numbers, positive or negative, whole numbers or decimals, fractions can be placed on. (TT3)

- Those (7% of the full sample) who provided an indication of how it could be done and attempted, though none successfully, to do it. TT33, TT39 marked representative numbers on, accompanied by some explanation:

> [Drew line and marked -4 at the left end, 0 in the middle and 0.4 on the right of zero] Yes - turn fractions to decimals (but not π - irrational) (TT33)

> Would be possible, but the line would not be very comprehensive. [Drew a line and wrote -4 at the left end and 9/10 at the right] (TT39)

TT48, TT66 marked a number line with a focus on ordering all the numbers, (apart from the square roots), but gave no indication of unit size and relevant partitioning:

> All of them can be put in one line. I thought of putting them all in decimal form and then placing them in order [Drew a line with integer intervals from -4 to 1 and - and + at either end. Divided the 0 to 1 interval into ten subintervals and marked 0.1, 0.2 (1/5) and 9/10] (TT48)

> Start with + and -, find smallest number to put at left. Disregard $\sqrt{\pi}$ as it is a continuous decimal and can't be placed on the number line. Find largest number and put on the right. Place all other numbers in between, after adding 0. Not sure about $\sqrt{2}$. (TT66)

TT41 was quite explicit that she only guessed the size of the relevant interval:

> Deciding what is bigger than what and guessing at the size of the intervals [Drew a line and tried to put all the numbers on, but the scale used was wrong] (TT41)

whilst TT32 focussed first on order without construction of the line and then attempted to decide where each number would go:

> [Without drawing a line, wrote the numbers in the following order: -4, 0.1, 0.2 (1/5), 1/2, 5/8, 9/10, 1, $\sqrt{2}$, $\sqrt{\pi}$] Try to order the numbers and decide which point on the line they occur. (TT32)

The evidence suggests that the greater majority of the teacher trainees had difficulty recognising number lines presented in an unconventional way. As we saw in §5.2.1, the general impression gained from the trainees' conception of the number line was a line containing whole numbers. Additionally, the fact that these students did not recognise that the lines within Figure 5.2 could be merged, suggests that they have no concept that notion of continuity underscores the number line. In some instances, the students were not able to recognise the meaning associated with $\sqrt{2}$ and $\sqrt{\pi}$ and made comments such as:

> What is the $\sqrt{2}$ and $\sqrt{\pi}$? [Line 3]. ... What is $\sqrt{\pi}$? [Line 5]. (TT41)
>
> How can this be represented on a line? π is an irrational number [Line 3]. ... Easier if $\sqrt{\pi}$ was expressed as a number [Line 5]. (TT43)
>
> You can't put π on a number line as it is not known what the exact number is - irrational number [Line 3]. (TT33)
>
> (circled the 1 and the $\sqrt{\pi}$) Is there any relation between these numbers [Line 5]? (TT39)
>
> (circles the $\sqrt{\pi}$) Scary! [Line 3] I would not give any of these number lines to children! (TT51)

The fact that 50% of the students did not recognise that the lines within Figure 5.2 could be merged suggests that their notion of continuity may be associated more with perception than with conceptual understanding. Thinking about the number line in a more general sense, as required when asked to give a definition, presents them with a more difficult task that is resolved by talking about a specific line or referring to something that they can do with the line.

It was relatively unfortunate that the way in which these teacher trainees would apply their number line knowledge within the classroom could not, for the reasons outlined within §4.4.1, be pursued further. However, their conceptions of the number line can be placed within the context of the practising classroom teachers whose lessons were observed during this study. It is to their perceptions and use of the line to which we now turn.

5.3 Practicing Teachers' Perception of the Number Line

The teachers who taught the seven classes that were observed were each informally interviewed about their conceptions of the number line and two main issues were raised:

- Whether or not they thought that the number line was a good representation of the number system (§5.3.1)
- When and how they used the number line (§5.3.2)

5.3.1 Teachers' Perception of the Number Line as a Representation of the Number System

3 of the 5 teachers identified the number line as a good representation of the number system because it carried the very ideas that 42% of the trainee teachers expressed with their definitions of the line. That is an emphasis on order and sequence:

> Yes! I suppose it is because it is natural order in a sequence, isn't it? (Y2 Teacher)

> It's a good representation for them to be actually able to see it! It has it (numbers) all in order and they can see it! (Y5 Teacher)

The fact that students could 'see' the number line was one of the reasons why a teacher teaching numeracy to Year 4 thought the number line was a good representation of the number system

> Because it's visual and children like visual things, and they can come up and interact with it. (Y4 Teacher)

Having something to see enabled some of the teachers to be quite specific in talking about the number line although there was evidence that this could lead to the sort of confusion identified by Skemp (§2.4.4), particularly if we recognise the hundred square as a segmented number track:

> I have got the number line, which is really useful, but because it's so long, it is quite hard... It's at least two metres (a number line on laminated card under the board). I do

> refer to it quite a lot, but I do use the number square as well. I do try and encourage the children that it's the same. (Y2 Teacher)

This similarity between the hundred square and the number line was also volunteered by the Year 3 teacher. He indicated that the number line is a good representation of the number system when used to develop subtraction, but not so easy as the hundred square which is

> easier than sometimes using the number line. Really, they're sort of similar things, but this goes zero to one hundred, this goes from one to one hundred, so it's the same really... (Y3 Teacher)

Other evidence associated with seeing and with accessibility came from the Year 1 teacher, who when asked if there was a difference between a number line and a ruler, replied:

> I just use the ruler, because it's a good individual tool and easily accessible. So if they want to use the number line it's immediately accessible. (Y1 Teacher)

Within her teaching of the classroom lessons, this teacher and the Year 2 teacher both drew an analogy between the number line and the ruler:

> A ruler is a bit like a number line. (Y2 Teacher)
>
> The number line here is like a ruler. Use a ruler[7] as a number line to help you. (Y1 Teacher)

However, the Year 2 teacher preferred to use the hundred square

> I do use the hundred square as well in the classroom, coz that's easier to display to be honest. (Y2 Teacher)

[7] The ruler the teacher referred to and given out to the children was one that represented a number track. It was a wooden 30cm stick divided in squares, with the first coloured yellow, the next green, the next yellow and so on and so forth. Within each box a natural number was written, starting from 1.

One of the teachers explicitly thought the number line was a good representation of the number system, because of the arithmetic that could be done with it:

> Yes! Very good! Use it to bridge through multiples of ten. Partition the numbers and then the tens and then the units, if they're doing addition. And if you're working out subtraction. (Y3 Teacher)

The teacher teaching Year 6 among other classes was the only teacher who gave a response that made any reference to the fact that the number line (although finite in her terms) was a representation of the number system:

> ... I think Year 6 children are quite good to see that the number line represents quarters, halves, numbers up to a thousand or even negative numbers.

This teacher's response to the question "Is the number line a good representation of the number system?" bore remarkable similarities to the trainee teachers' conceptions of the number line. Two of the five teachers referred to pattern, order and sequence. There was reference to the number line as tool, but only one reference to the variety of numbers that could be represented on it. However, whilst all of the teachers could talk about what it may look like or what it may be associated with, none provided a sense of its continuity and density. Those teachers who referred to the hundred square or to the ruler did not make a distinction between the abstract nature of the number line representing continuity and the more concrete nature of the alternatives that represented the discrete nature of number.

We now turn to consider how these teachers indicate the ways in which they used the number line within the classroom.

5.3.2 Use of the Number Line

The teachers were asked to indicate when and how they use the number line. Three themes emerged from the discussion; developing skill in recognising the order of numbers on the number line (Y1 Teacher), using the number line as a tool to resolve numerical procedures (Y2 and Y3, Y5 Teachers) and an implicit rather than explicit expansion of the number system (Y3, Y5, Y6 Teachers).

The Year 1 teacher indicated that recognising the order and sequence of numbers was the main use that she made of the number line:

> [I] use it at least twice a week. During a numeracy session or the odd five minutes you spare. The number line resource it's just there. For example when sitting on the carpet, I might say close your eyes and I'll take a number off and ask them what number is missing and find what comes before and after. Or mix numbers zero to ten and get them to re-sequence them. (Y1 Teacher)

A focus on the use of counting, forwards and backwards was the issue for the Year 2 teacher, whilst the Year 3 teacher combined a focus on the processes of addition and subtraction with conceptual reference to the notion of multiplication and fractions:

> When doing number sequences circle the numbers and look at the steps we're taking each time. If it's addition, look at it with jumps. Subtraction counting back. (Y2 Teacher)

> Use it for addition, subtraction mainly. Maybe some multiplication as repeated addition... For fractions, labelling the ends zero and one or zero and two. For reading scales, labelling divisions, marked scales, measurements. (Y3 Teacher)

We may see the reference of the Year 3 teacher as an indication that the number line is being used to expand the range of the number system, but there is a sense from his response, and from the responses of the Year 5 and the Year 6 teachers, that any reference to a range of numbers that may contribute towards a conceptual understanding of a number system beyond the counting numbers is limited. In each instance where fractions, or negative numbers are mentioned there is an overwhelming sense that each item is seen as discrete and not part of a unified whole that may see fractions being inclusive of whole numbers as rationals, or negative and positive numbers being seen as the set of integers:

> Use it for so many different things. Negative numbers, subtraction, addition.
> (Y5 Teacher)

> Use it for place value, fractions, whole numbers, negative numbers and for children to come and interact with me. (Y6 Teacher)

What is particularly interesting is that the number line is seen as something which is concrete, in the sense that a number can be "taken off" or "circled" or that we can "jump" along it. What is essentially an abstract conception, for pedagogic purposes is concretised so that it may be used as a tool — simply a ruler!!!

5.4 Chapter Summary

Within this Chapter, we have examined conceptions of the number line of teacher trainees and practicing teachers. It was noted that Herbst (1997) provided a definition for the number line, an indication of how it may be constructed and made reference to it as a "helping tool". His definition indicated the successive translation of a defined unit that can be partition in an infinite number of ways. The greater majority of the teacher trainees and the teachers were able to allude to one of these features, if we accept that the notion of partitioned can be equated with the notion of 'equally divided', but none gave the impression that their definitions included all of Herbst's features. Indeed, any reference to infinity and partitioning the unit was extremely limited. Instead, the majority of trainees and the teachers appeared to either focus on descriptions that may fit particular, or in some cases a generic number line, or they provided a 'definition' that focussed on its use as a helping tool. The language used to define the number line was either associated with specific descriptions, for example:

> A straight line with equal distances marked. (TT7)
>
> I have got the number line, which is really useful, but because it's so long, it is quite hard... It's at least two metres. (Y2 Teacher)

or with description associated with episodes.

> A piece of apparatus with equal divisions which children use to help them count. (TT10)
>
> The number line here is like a ruler. Use a ruler as a number line to help you. (Y1 Teacher)

Descriptions and episodes were frequently associated with terms such a "pattern", "order", "row" and "sequence":

> An ordered set of numbers in sequence, horizontal. (TT6)

I suppose it is because it is natural order in a sequence, isn't it? (Y2 Teacher)

Conceptions associated with the idea that the number line was an abstract representation of the number system were absent from any discussion with teachers or from analysis of questionnaires from the teacher trainees. Indeed conceptions of continuity amongst the students were only indicated when they considered a visual representation of a segmented number line. Most perceptions, particularly from amongst the teacher trainees, seemed to imply that number lines only contained whole numbers. Several trainee teachers did make reference to aspects of the number system wider than that of whole numbers, in particular reference to fractions, but the sense was that these were discrete number systems and not an integral part of a whole system.

Teacher trainees who thought of the number line in more sophisticated ways, tended to agree that all lines could be merged into one and at the same time attempted to put them on one line. Out of the whole sample only two students (TT48, TT23) expressed conceptual understanding of the number line either through their comments upon specific lines (from within Fig. 5.2) or by evoking the notion of infinity when defining the representation. Overall, it seems that students who provided descriptive responses when asked to define a number line (TT15, TT20, TT61), also gave descriptive responses when they were presented the lines within Figure 5.2. It could also be argued that those students who think of the number line as a tool are unable to effectively respond to merging the lines together (TT10, TT34, TT62, TT65, TT52).

The impression left suggests that teacher knowledge of the conceptual features associate with the number line is extremely limited and its benefits are pedagogical in that it is a helping tool that supports the development of children's operational skills. The children's conception of a number line identified within the pilot study indicated that what could be done with a number line was the key feature they associated with it. We now turn to see what focus is given to the number line within the classroom and what it is that children construct from their activity.

Chapter 6: Teaching and Learning with the Number Line

6.1 Introduction

Chapter 5 indicated that teachers and teacher trainees defined the number line in terms of descriptions of particular lines and though they talked in terms of equality, order, sequence and pattern, they placed an emphasis on its use in applying procedures. A small minority saw little difference between it and a hundred square and therefore appeared to be missing the point between the notions of discreteness and continuity that identify the differences between these two representations of number. Most seemed to reflect upon the number line as a concrete representation, but failed to make the link between the abstract counter-part that replicates and translates a chosen unit, is infinite in magnitude and can reflect the subdivision of each unit into an infinite number of partitions.

Given that these findings were drawn from teacher responses to a questionnaire and informal discussions with individual teachers, this chapter now considers the way that teachers use the number line in their classroom and provides an indication of children's interpretation of this use.

Two themes guide the discussion:

- Teacher indications of the meanings associated with the number line, with particular reference to whole number, fractions, decimals, and children's interpretations of these.
- Teacher indications of the way the number line may be used and children's practice and interpretations of this use.

Since the chapter is associated with the ways that teachers used the number line, the data collection was through lesson observation and subsequent to each lesson discussion with the identified children (§4.4.3) to establish what they thought they had learned during the lesson.

A total of 23 lessons were observed across the six year groups Y1 to Y6 over a five-month period from the beginning of September 2003 to the end of January 2004. Each of the teachers had agreed to allow observation of lessons where the number line would feature and because of this different numbers of lessons were observed for each year group. This ranged from seven lessons within Year 4 to two lessons within Year 6. Four lessons were observed within each of Years 1 and 2, three within Years 3 and 5. While lessons within Year 1 were observed, there was no follow-up questioning with the children but lessons placed a context on the initial development of the number line and therefore are reported. An analysis of the content of the lessons indicates that at least two lessons within each year group apart from Year 6 were associated with the use of the number line in the context of whole numbers. Whilst this was always the case within Years 2 and 3, within Year 4 the remaining lessons were associated with the use of the number line and fractions and within Year 5 the number line and whole number. The lessons within Year 6 were associated with fractions, decimals and negative numbers.

For each lesson observed, a plan was available and lesson observation focused on critical incidents that included:

- references the teacher made to the number line.
- examples provided by the teacher.

To reflect the emphasis given within these lessons, this chapter is structured to consider the use of the number line in the context of whole number, its use in the context of fractions and its use in the context of decimals. During each observation, an assessment was made of external representations of the number line that were apparent within each classroom.

6.2 The Number Line and its Use in the Context of Whole Number

Half of the observed lessons considered the number line within the context of whole numbers. These ranged from ordering number on a number line (Year 1, §6.2.1) the use of the number line in elementary addition and subtraction (Years 2, §6.2.2), through to the addition and subtraction of two digit numbers with the number line (Year 3, §6.2.3), the revision of the approaches used for two digit addition and subtraction and an extension to three digit addition and subtraction (Year 4, §6.2.4). Within Year 5 this was reinforced through the use of partitioning and led to the multiplication and division of numbers by 10 or 100 (Year 5, §6.2.5) and within Year 6 we see the use of the number line in the context of division (§6.2.6).

6.2.1 Initial Considerations:The Number Line and Number Order (Year 1)

To place the later discussion in context we first consider presentation within Year 1. The teacher's lesson plans were printed from the internet and were part of a generic plan prepared by the local authority and recommended for schools in its immediate area. The objective for the first of the observed lessons was to recognise numbers and place them in their appropriate order on a number line. In this context the teacher used a segmented 0 to 20 line with each whole number identified and below it an unmarked number line with the intervals represented by dots with some numbers included but others missing. Counting from zero, the children were required to complete the sequence by selecting and inserting the missing numbers. Immediately after this, the teacher referred to a 'chart', diagrammatically presented within Figure 6.1.

100	200	300	400	500	600	700	800	900
10	20	30	40	50	60	70	80	90
1	2	3	4	5	6	7	8	9

Figure 6.1: Diagrammatic representation of the Year 1 teacher's representation of the number system

Pointing to nine (red), the teacher indicated:

And that's the number nine. You've got your number line. (Y1 Teacher, Observation 1)

Additionally during this lesson, the teacher made use of a 100-bead string and partitioning it into 20 and 80 emphasised forward and backward counting in ones from one to twenty. Zero was identified as a remaining piece of string before one was counted.

As a final activity, the teacher used an unmarked line and making use of randomly distributed numbers asked the children to make their own number lines by placing the numbers in order. The unmarked line was later introduced as an empty number line.

> We've got a really, really, really super duper exciting activity today. You're going to make your own number lines. You've got an empty number line. Empty number lines don't help us do they? (Y1 Teacher, Observation 1)

During the second observation, the teacher introduced the lesson saying:

> Right children, today in our numeracy, we are going to be looking at numbers on a number line. If you look against my board there, you can see *a big number line on the top that goes from zero all the way up to ten and then a smaller number line underneath that goes from one to twenty.* We're going to be working with those numbers and see how to write them, how to read them and also how to order them. By ordering them, I mean to put them in the right order. Starting from zero, going all the way up to twenty, or start at twenty going all the way back. (Y1 Teacher, Observation 2)

It is worth noting that the teacher referred to both a number line and a number track as a number line.

The notion of order was reinforced in subsequent lessons through the notions of 'before' and 'after':

> We're looking at our number line. We're looking at the words 'before' and 'after'. [Teacher points to a number on a number line and asks about the number that comes before and which number comes after]. (Y1 Teacher, Observation 3)

In a subsequent lesson, the teacher reinforced her activity to place missing numbers on the number line. The activity was supported by the children's recitation of the number sequence, but the children's difficulty in selecting the appropriate number symbol to place within the empty space was seen as an inability to recognise the meaning associated with the number symbols:

> The trouble is that they're not recognising the numbers anyway on the number line, so it is hard to put them on the right spot. They can count, but they can't recognise them.
> (Y1 Teacher, Observation 4)

A later activity involved placing numbers in sequence on the floor, each number representing a 'stepping stone'. The teacher referred to this as a number line, but also noted that in finding the position of particular stone there was a tendency for the children to count from one:

> Did you notice? When Tommy went from fourteen back to thirteen. He started at the beginning. They've got to count on zero, one, two, three, ... He started counting because his brain knew they were in order. When you learn number lines you know that they go... the numbers go in order, but they count. They knew that they count one, two, three, four, five and so on. So he knew that the number we landed on was the number I needed, but he didn't know that that was thirteen. What we need to teach is that our number line goes along counting in order to do anything else. (Y1 Teacher, Observation 4)

This series of lessons gave an insight into the emphasis placed on the number line in the children's first full year of school. Although the essence of the activities was forward and backward, counting associated with ordering of the numbers to 20, the number line was used as a representation to support this. However, the term was used in such a way that it could refer to an ordered sequence of the numbers or as a sequence of discrete numbers as seen in the stepping-stones episode. Thus, very early in their mathematical development, the children experience the notion of number line associated with the number track whist the teacher was frequently ambiguous in her use of terms whilst also referring to the hundred square as an alternative "tool" that would support the children's development in counting and identifying particular numbers:

> If you need help, you can look at our hundred square or number line.
>
> (Y1 Teacher, Observation 4)

Although there were examples of the use of a marked line (with no numbers), there was no reference to the notion of repeated unit. The emphasis continued to be placed upon order and counting. Difficulties associated with this ordering were identified as being associated with the children's inability to recognise the number symbols.

Although an initial attempt was made to talk to the children about these lessons, this was aborted since they were generally unable to articulate their understanding.

6.2.2 Introducing Addition and Subtraction (Year 2)

Following a lesson during which the class teacher's objective was to emphasise forward and backward counting with an emphasis on the phrases 'one more than', 'one less than', 'ten more than' and 'ten less than' (observation 1), the teacher proceeded to reinforce this (observation 2) before indicating:

> Now we are going to be using number lines to do some adding on.
>
> (Y2 Teacher, Observation 2)

At the start of the activity phase of the lesson, the teacher used an empty number line in demonstration mode to add 11 and 37. Although the notion of empty number line had been referred to in the previous lesson, and this was identified as simply a line with neither numbers nor marks on it, there was no explicit explanation during observation 2 that explained its relationship with the segmented lines the children had been using previously. The teacher proceeded to place 37 near the left hand side of the line and with her index finger, the teacher traced a jump of 10 and then a jump of 1, stressing the partition to demonstrate how children should 'add ten and then 1' and thus starting at 37, 'jump ten' to 47 and then 'jump one' get the sum 48. After a second similar example, the teacher concluded:

> This is the way we can use our number line to help us add numbers.
>
> (Y2 Teacher, Observation 2)

The numbers used were beyond the range of those that the children had experienced within the previous observations and made use of the empty number line without any specific association being made to a segmented number line.

Prior to a period of reinforcement, during which the children were to practice adding single digit numbers to give a sum below 20, the teacher stressed that the numbers can be turned around so that the bigger number could be placed on the number line first and it was added that:

> You're going to use a ruler as a number line to help you answer the questions.
>
> (Y2 Teacher, Observation 2)

During the third observation, this theme was developed further with the teacher's objective for the lesson articulated as:

> ... to understand that addition can be done in any order, that means we can swap them around, but it is easier to just put the largest number first. This is because it helps us to work out sums such as 4+84, 5+57 quicker. (Y2 Teacher, Observation 3)

After an introductory activity within which a string with 100 beads was used to count on a particular amount from a nominated start, the teacher introduced three other representations, a long plastic card on which the numbers 0 to 20 were ordered, a hundred square and a 0 to 100 number line. The plastic card was also referred to as a number line. These representations were used interchangeably to complete number combinations such as 12 + 5, 15 + 3, etc. Each time the teacher started with the larger amount, always emphasising that it was easier to start with the biggest number, and demonstrated the addition of the smaller amount by counting-on and referring to 'jumps'.

At the start of the individual activity phase, the teacher distributed a green ruler to each child explaining:

> A ruler is a bit like a number line. (Y2 Teacher, Observation 3)

During the third observation, the children were required to find the sum of number pairs such as 12 + 4, 4 + 12, 86 + 4, 4 + 86, etc. She added:

> Since the ruler has not got all the numbers, you can use the hundred square or the big number line [0 to 100 under pin-board] or use your ruler or try to do it mentally.
> (Y2 Teacher, Observation 3)

The objective of observation 4 was specified as subtraction. Terms such as 'take-away', 'minus' and 'subtract' were used to interpret symbolism such as 8 – 3. To find the solution the teacher referred to a 'washing line' similar to the one used in Year 1 and starting from 8 demonstrated a count-back procedure using successive 'jumps' of one until three had been counted. She then continued by demonstrating how subtraction may be completed on an empty number line by using the example 27 – 11. After drawing an empty line, 27 was written on the right hand side and a right to left jump of

10 was demonstrated to reach 17. A further jump of one enabled her to mark 16. She stressed that the take-away jump of ten must be bigger than the take-away jump of one and concluded by saying:

> When you start seeing it like this, you can see what's happening with the numbers [and pointing to the washing line continued] coz this is like a ruler, the number line here isn't it?
>
> (Y2 Teacher, Observation 4)

During the first observation, the teacher had referred to a row of numbers as a number line and the ambiguous way in which representations were given equivalent status was exemplified during the further observations. During observation 4 the teacher used three representations; a laminated 0 to 100 number line with only the tens marked, a permanently displayed 0 to 100 line under the blackboard and a laminated number track marked 0 to 20.

With reference to the 0 to 100 number line, the following exchange took place between the teacher (T) and the children (C)

T: Has it got all the right numbers on it? From zero to one hundred?
C: [Some said] 'yes', [others] 'no'.
T: Has it got them all on it?
C: No!
T: So has it got one, two, three, four, five, six, seven, eight, ... ninety-eight, ninety-nine, one hundred?
C: No!
T: What has it got on this number line?
C: [One child responded] It has got multiples of ten on.
T: So it's got zero, ten, twenty, thirty, forty, fifty, sixty, seventy, eighty, ninety, one hundred [said together with children]. But this number line [pointing at a number line from 0 to 100 in ones, under the board] has what on? Just the multiples of ten? Has it got some numbers to a hundred or has it got all the numbers to a hundred?
C: All the numbers to one hundred.
T: So this one has got the multiples of ten, this one here has got all the numbers from zero to one hundred.

The teacher then turned to the number track (with zero marked in a space)

> This has got ten, it's got zero first of all, it's got my multiple of ten and it's got my multiple of twenty look. So this is the whole number line from there (zero) to there (ten). But there's lots more numbers on it isn't there, them there [pointing to the multiples of ten line] is on there [pointing to the 0 to 20 track]! Coz this has got every single number from zero to twenty. (Y2 Teacher, Observation 1)

The ambiguity was further compounded with the statement that:

> The hundred square is like a number line, but with one bit on top of the other.
> (Y2 Teacher, Observation 1)

This ambiguity was later expressed through the interviews with the children. When asked what they had been doing in particular lessons, the children indicated:

> Put numbers on number lines and hundred square... counting in tens. The teacher takes out a number of the hundred square and we had to find which is missing. (Child 2.1)

> I had to do some work, I had a playtime... I had to put the numbers in the right order in tens.... Used a ruler... all the way up to one hundred. (Child 2.3)

> I was taking-away and multiplying... [had to] guess where the numbers went on the hundred square. (Child 2.4)

> Using a number line (means hundred square), coloured it with pens, got numbers on it and the number line don't go up to one hundred, but the square does. (Child 2.2)

Interestingly, though four representations were used within the classroom, a segmented number line, a number track and a hundred square, and a ruler as a number line, it is the hundred square that dominates the articulations of these children. Child 2.2 was the only child able to make some distinction between the track and a number line. When given a number line to talk about, she indicated:

> This is a number line. A real number line. There's nothing. Nothing's between one and two because nothing's there. (Child 2.2)

The evidence from this series of observations indicates that there is some considerable ambiguity in the use of terms such as number line, number track (washing line) and hundred square. The Y2 teacher had, during the informal discussion, indicated that she believed the hundred square and the number line were the same thing (§5.3.1), but during her interview she made no reference to the use of a washing line (as a number track). The consequence is that the conceptual differences between the number line as a representation of continuity and the number track (together with the segmented track that is the hundred square) as a representation of counting numbers were missed by the children. The most sophisticated child's response regarding their difference was given by Child 2.2 but it was a difference articulated from the perception that there is a gap between the positioning of the numbers on the number line "because there is nothing there".

The emphasis of teaching was clearly on the use of the number line as a tool to order numbers, to develop counting and support procedures for addition and subtraction. This use was accompanied by the use of rules such as "put the largest number first" and followed by left to right jumps for addition and right to left jumps for subtraction. However, the emphasis on the number line as a tool became much more apparent during observations of the Y3 to Y5 children.

6.2.3 Two-Digit Addition and Subtraction with the Number Line (Year 3)

Though there was an emphasis on practicing addition and subtraction within Year 3, during the first observation a part of the lesson focussed on ordering numbers by pinpointing position on a number line. Initially this was on a number line segment marked 0 and 10 at each end. To develop an approach for finding the position of magnitudes between 0 and 10 the following development took place:

T: So on this side we've got the number zero and on this side we've got the number ten. So what are we counting in?
C: Ones.
T: Let's count in ones then.
C: Zero, one, two, three, four, five, six, seven, eight, nine, ten.
T: Who knows what will be here (pointing at different points on the counting stick).
C: Five
T: How did you know so quickly?
C: Five is in the middle between ten and zero.
T: What would be here if five is here?
C: Six.
T: So started from five, counted one more, go up to six. If five was here what would be here?
C: Four.
T: And here?
C: One.
T: Here?
C: Nine.
T: Here?
C: Six.
T: That's a difficult one. Five was here, you got six, then seven. Count on from five, count back from ten.

Two things are noticeable from this episode. The first is the strategy that the children explain for identifying the first magnitude — that of five. The children's approach was to associate five as being in the middle of the 0 to 10 line. The second consideration is the development that uses this position to identify the immediately preceding and immediate subsequent numbers. However when seven was pinpointed, the children claimed it was six (because it came after 5?). Note however, that after identifying the five they were given two further approaches — 'count-on' from five and 'count-back' from 10. First, apply a strategy and then use the outcome of the strategy to count. This is an issue that we shall return to in Chapter 8 (§8.3).

Subsequent observed lessons focussed on the use of the number line for addition and subtraction. In general, the approach for addition made explicit counting up in ones whilst that for subtraction was counting back in ones. Subtraction was seen to be the opposite of addition. The dominant representations used were a number line and a hundred square and as we have seen earlier within Y1 and Y2, these two representations were regarded as similar, the difference being simply the presence or otherwise of zero:

> Really, they're sort of similar things, but this goes zero to one hundred [number line], this goes one to one hundred [hundred square], so it's the same really.
>
> (Y3 Teacher, Observation 2)

This second observation started in a similar way to that of observation 4 within Y1 — clarifying words that could mean subtraction and emphasising procedures of going forwards or backwards dependant upon whether or not the children were doing addition or subtraction. However, the Y3 teacher also associated these actions with actions on the hundred square:

T: This week in numeracy we are going to learn how to do subtraction. Tell me different words for subtraction.

C: Decrease, take-away, differences, etc.

T: If I'm taking-away which way am I going on the number line? Am I going forwards or backwards?

C: Backwards.

T: If I'm taking-away ten on the hundred square am I going down or up?

C: [One child] Down!

T: The largest number has to come first on a subtraction. We are going to use this number line to help us. Twenty-two take-away two. Come to the board and use the number line — take-away two and you move back.

After this example, two others were presented 33 – 9 and 46 – 8. The teacher then reinforced his explanation of the difference between the hundred square and the number line and provided what was thought to be an advantage of the latter:

> Why is using the number square easier when we're subtracting bigger numbers? Easier than sometimes using the number line. Really, they're sort of similar things, but this goes zero to one hundred, this goes from one to one hundred, so it's the same really... but why is it easier? Coz when we want to count the multiples of ten, we just move up a square, but in this one [number line] we have to count the multiples of ten.
>
> (Y3 Teacher, Observation 2)

During a subsequent lesson the addition focus changed into bridging through a multiple of ten, demonstrated through the addition of 35 + 8 and through the use of the empty number line:

T: What is the first thing I do after drawing the number line?

C: Write thirty-five at the left end.

T: Why do we start with thirty-five [and not eight]?

C: Coz it's the biggest number.

The process continued by counting on 5 to make 40 and then counting on another 3 to make 43. Interestingly, the teacher then said:

> I'm not interested in any answers. You can get it wrong, but as long as the method is right, that's what I'm interested in. (Y3 Teacher, Observation 3)

The children's reflections on these lessons indicated that:

> The teacher done a sum. He had to jump on a number line to get the answers. (Child 3.4)

> Counting forwards and backwards with numbers on a hundred square. (Child 3.3)

Whilst one child could be fairly explicit about a procedure:

> We had to find the difference between the take-away sums. Thirty-nine take-away thirty-six are close together, so you get a number line or a hundred square and count down to the highest number. (Child 3.2)

Others had forgotten it:

> No! [I can't tell you what the lesson was about]. I always forget stuff... number line.
>
> (Child 3.1)

There was evidence of operational difficulty and confusion. When Child 3.4 was asked to find the difference between 18 and 14, she said:

> It's got the four at fourteen... and fourteen... is got the eight at eighteen. The difference... the four has a line. That is the difference between two numbers.... I don't understand!
>
> (Child 3.4)

The children's conceptions of any differences between the number line and the hundred square reflected the comments of their teacher but gave only part of the difference:

> [A hundred square] is a number line coz it's got the one that goes up to a hundred.
>
> (Child 3.3)

Once again, the observations within Year 3 provided support for the evidence obtained from within the earlier years. There was ambiguous use of the number line and the hundred square, the procedural emphasis for addition and subtraction supported their inverse nature (there was no reference to count up as an alternative choice for subtraction) in that for addition the largest number to be added started on the left and then count-on was used, whilst for subtraction the largest number started on the right and then count-back applied. There was no conceptual reference to the differences in the two forms of representations. During the first observation, a strategic approach to pinpointing a particular magnitude on a number line segment 1 to 10 was considered. After finding the middle number, 5, the children then counted.

6.2.4 The Number Line and Three-Digit Subtraction: Year 4

Though seven lessons were observed within Year 4 only two, observations 6 and 7, focused on whole number arithmetic. Though the emphasis was, the subtraction of a three-digit number, observation 6 started with the use of bridging ten to solve the subtraction of a two-digit number. An empty number line was the main resource. The general procedure was given to the children:

> Smallest [number goes in the beginning] coz the number line goes up... The first thing I want to do is to get to the next multiple of ten... You're looking on the nearest multiple of ten. (Y4 Teacher, Observation 6)

This was then followed with the example of subtracting 37 from 43:

T: [Solving subtraction on a number line] Does anybody know how to do this sum by counting on? If it was thirty-seven there [left end] what's the number at the other end?
C: Forty-three
T: The first thing I want to do is to get to the next multiple of ten. The next one after thirty-seven. What is it?
C: Forty.
T: What have I added on thirty-seven?
C: Three.
T: What have I added to forty to give me forty-three?
C: Three.
T: So what's the answer? Three add three.... So therefore, if you take thirty-seven from forty-three, then you have what left?
C: Six.
T: That means you've got forty-three sweets and you give out thirty-seven to your friends, you will have six left. Does anybody know what this line is called?
C: Number line.

Within observation 7, the procedure for resolving the solution to subtract a three-digit number was considered. However, the lesson started with reinforcement of the above lesson but this time the teacher did not use a number line but simply used an arrow representing 'jumps' to connect numbers on the whiteboard:

T: We're going to look at methods of taking-away, using the number line. Let's start with an easy one. Ninety-five take-away eighty-six. We're going to put eighty-six here [without drawing a line the teacher wrote 86 on the board] and we want to get to ninety-five [written on the right hand side away from the 86]. So what jump should we do first?
C: Four.
T: And then jump?
C: Five.
T: So what's five and four?
C: Nine.

After the children had been given an opportunity to practice a similar approach using the number line with three-digit numbers, and where in fact there were many incorrect answers, the teacher concluded:

> Oh, dear! I don't think we're very good at this... What a lot of wrong answers?
> (Y4 Teacher, Observation 7)

When the selected children from within Year 4 were invited to work out the answers to addition and subtraction problems, only two (Children 4.3 and 4.2) attempted to use a number line and neither of these were successful in obtaining the correct answer. However, the reasons for the errors differed. Both children in fact executed a correct number line procedure that emphasised a focus on bridging the tens. For example, Child 4.3 attempted to use the number line to add 84 and 36, but in fact used a procedure to subtract 36 from 84 (recorded as in Figure 6.2):

> You start with the lowest number first [36 at left end of an empty number line]... and then you start with the biggest number after [84 at the right end of the line]. We're doing jumps, ... half jumps like that and we had to add... quite to the nearest ten. Forty. And then... six, seven, eight, nine, ten... you added four there to get ... forty to eighty... you had forty-four... add four... and then it equals forty-eight. No, that would... I was meant to put take-away (-) there [she then changed the plus sign of 84+36 to read 84–36].
>
> (Child 4.3)

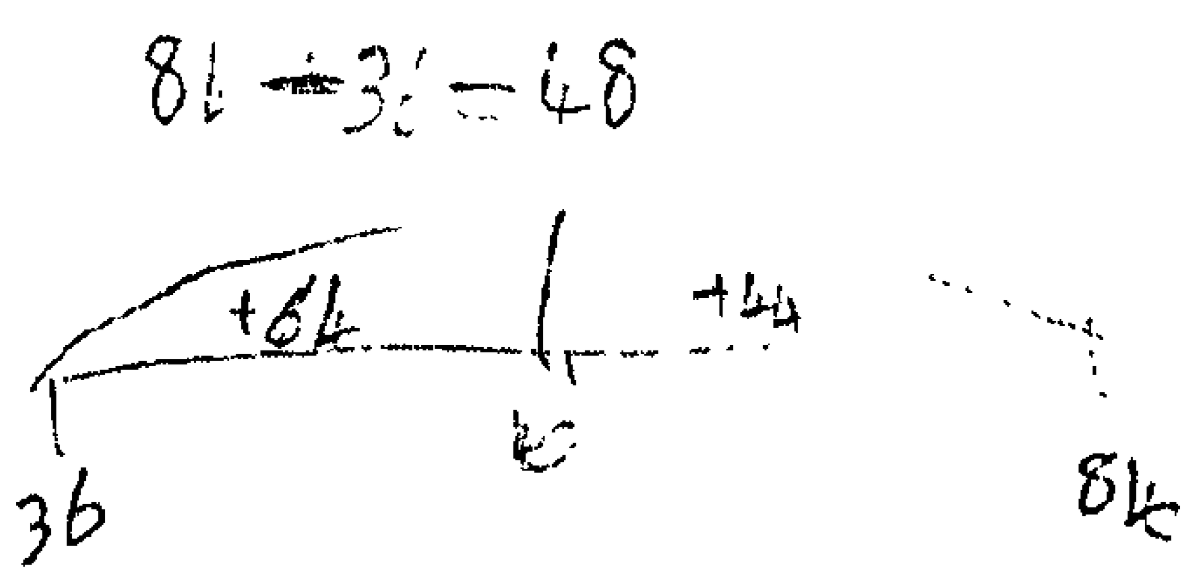


Figure 6.2: Performing subtraction instead of addition (Child 4.3)

It is noticeable that the child's marks as on Figure 6.2 are not clear. Initially she marked 6 and then crossed this through to mark 4. This gave her the forty (although the middle mark is not a clear 40). She then marked the 44 and adding 44 and 4 gave her 48. The child had assimilated the exact procedure used by the teacher during the lesson but recognising her error, she casually changed the sign of the addition from positive to negative.

Child 4.2 on the other hand was asked to obtain the solution to 503 – 103 and whilst the procedure he applied using number line 'jumps' (but as presented by the teacher no representation of the line) was correct, he then used a vertical algorithm to add the components of his 'jumps' but applied this incorrectly (Figure 6.3).

I put one four three (143) here and then I jump... I get seven, I get three or fifty and I add it... if we get, that... three hundred. [503 written to the right and the jumps and their result written between 143 and 503]. (Child 4.2)

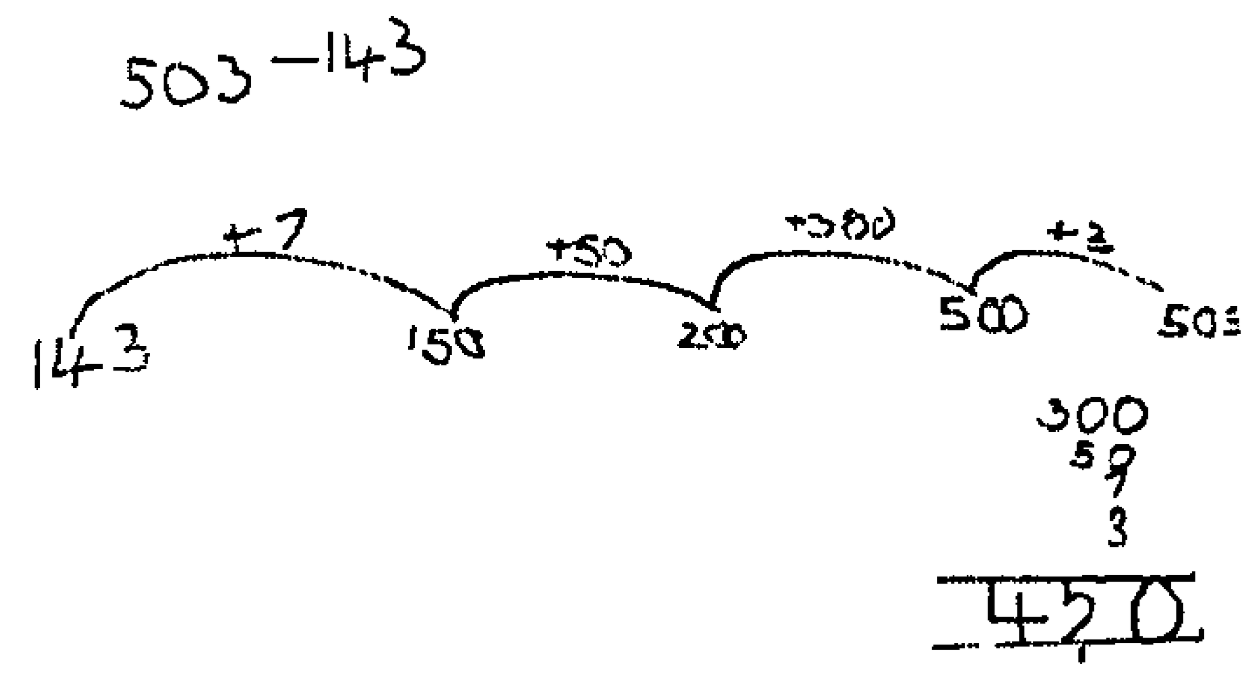


Figure 6.3: Performing subtraction without drawing a line (Child 4.2)

As can be seen within Figure 6.3, after carrying out the addition procedure to establish the sum of the jumps, Child 4.2 obtained a solution of 450 rather than 360. Here there are place value issues but that is a different problem. This child was asked why he did not draw a number line and only illustrated the jumps. He explained:

Coz I don't like drawing lines... what's the point of drawing lines... that's where you make a mistake when you draw out lines. In Year 3, coz I used to get told off... drawing lines. Coz I was doing it like that [draws line] and I was doing jumps like that [draws tall and narrow jumps on his line]. (Child 4.2)

It was not unusual for children to become confused when bridging tens and perhaps partly in recognition of this confusion initially applied the well practiced counting procedure to add or subtract. Child 4.4, for example, when asked to use a number line to solve 52 –17 drew a line with 17 at the left hand side and 52 on the right hand side. She then proceeded to count on from 17 making small marks until she reached 52. When asked by the interviewer (I) how many did she count she replied:

Child 4.4: Thirty-two, thirty-three, thirty-four, thirty-five.
I: Why up to thirty-five?
Child 4.4: Coz there were thirty-five spaces.
I: Where?
Child 4.4: In there. The little lines.
I: So how many lines were there?
Child 4.4: Thirty-five.

Child's 4.4 approach emphasised the use of counting unit marks that she had placed on the empty number line, but this did not always prove a suitable approach. She had some difficulty attempting to obtain the solution to 18 – 9 using an empty number line and recognising this she concluded:

> .... I know the answer. Nine... Because double nine is eighteen and eighteen take-away nine is nine. (Child 4.4)

Child 4.1 was invited to use a number line after indicating that she could not remember what was being taught during observation 7 and correctly obtaining the solution to 43 + 37 using partitioning (40 + 30 + 3 + 7). When she used the number line, she placed 37 on the left and 43 on the right. She initially added 3 to get to the next multiple of ten, 40, but then gave an answer of 46 (43 + 3). She realized something was wrong:

> Coz it ain't the same answer as when I partitioned it... I've done the number line wrong... The partitioning way [is better]. (Child 4.1)

Within Year 4, we see that in the context of whole number arithmetic, the teaching emphasis during the observed lessons was upon developing an approach to addition and subtraction and in particular bridging the tens or counting on in multiples of 10 or 100. This encouraged the children to solve addition and subtraction problems using the same approach, in essence using complementary addition (see Appendix II, Y3; 5.45). During the first observation, the children were encouraged to bridge through ten as a stepping-stone towards building the number to be added. In the second, the children were encouraged to a reversal of this process as the first phase in a process of building towards the difference between two numbers. What is particularly interesting is that the conceptual difference between taking-away (one number existing and taking an amount away through counting-back) and of finding difference (two numbers existing and counting-up the difference between the two (complementary addition) was not made explicit in any of the lessons observed within Y4 or the other classes. In essence whatever language was used, subtract, difference, minus, processes were developed through either count-up of count-back. However, as the lessons within Y4 progressed, the number line moved into the background but the use of jumps remained. Eventually the teacher gave the children the choice of whether or not they actually drew a number line — representing the jumps (as seen in Figure 6.3) became the focus of recording and discussion. Interestingly, though these observations were seen within Y4, they were actually associated with NNS outcomes of Y3 (Appendix II). The use of four digit numbers as recommended within the NNS was not observed.

It seemed that during the interviews children used the number line with some reluctance. Though they might use an empty number line and follow the indications of the teacher to "put the smallest number at the beginning and the biggest number at the end" (Child 4.1) to repeat "what we done this morning... half jumps like that and we had to add quite to the nearest ten" (Child 4.3), partitioning, but omitting the use of the number line or jumps, seemed better.

6.2.5 The Number Line and Arithmetical Operations: Year 5

Three lessons were observed within Year 5. One of these, observation 2, focused on the use of measuring with a ruler and this will not feature within the following discussion. The first observation focussed on multiplication and the third on addition. The discussion will start with the third since it draws parallels with what has been discussed with reference to Year 4 (§6.2.4).

The teaching objective for observation 3 included working out addition by partitioning, first using the tens and then the units, and then working out addition using the number line by first adding an appropriate number of tens onto the larger of the two numbers and subsequently adding the units of the smaller number to the total.

At the start of the lesson, the teacher presented the addition of 54 + 28 to the children and asked a child to explain how it was worked out. A child indicated that she would add 20 to 50 to give 70 and then add 4 + 8 so that 70 + 12 = 82. This would satisfy the first objective. The following exchange then took place.

T: Any other method?
C: ...
T: How about if you had one of these (draws a straight empty number line)?
C: A number line!
T: How would you use a number line to work it out?
C: ...
T: Where would I put fifty-four?

C: In the middle.
T: [Teacher wrote fifty-four in the middle] Then? [Teacher marks zero and one hundred at either end]
C: ...
T: If I add on ten where will I get to?
C: Sixty-four.
T: [The teacher then ascribes a jump of 10 to 64, then to 74 and then a jump of 8 to reach 82]. We didn't really need that bit [pointing to the 0 to 54 segment, so it was erased].

It is interesting that the child implied that her embodiment of the empty number line included the end points (although she did not make reference to them) so that she could suggest that 54 was in the middle of the line. This was an embodiment that was supported by the teacher's reaction, which was to mark the end points 0 and 100. There was no indication that the 54 could have gone anywhere on the line.

Following the exchange, the class were then given two further examples that were first solved using partition and then with a number line, but the activity presented to the children involved the addition of up to three-digit numbers using the standard algorithm. No links between this and the two approaches used at the start of the lesson were made.

After the lesson, the four selected children within class five were then asked to describe the substance of the lesson. In each instance, the children only referred to their use of the standard algorithm and when asked to give an example wrote the example as in the standard algorithm. Only Child 5.5 indicated that an alternative way of doing the addition was to use a number line. However, when she used the number line to add 462 and 276 she made two unsuccessful attempts:

(i) She initially drew a number line near the end of the page and wrote 276 at one end and 462 at the other end. Starting from the 276 she made a left to right jump of 200 and then identified a jump of -14 but unfortunately the jump continued in a left to right direction.

(ii) A second number line was drawn that covered the width of the page in case there was not enough room to put the numbers on.

This second attempt was not completed, but Child 5.5 indicated that:

> The vertical [algorithm] way is the easiest. The number line is the hardest. (Child 5.5)

The veracity of this statement was proven, for her, by the attempts she made to solve 12 + 39. After drawing an empty number line, she placed 12 and 39 at each end and after making disproportional jumps of 20 and 9 from the 12 gave the answer of 39.

Child 5.4 on the other hand could use the number line to confirm an estimation or to validate answers he had worked out mentally. Asked to give the answer to 76 + 49 he drew a line with the ends marked 0 and 150 ("It doesn't need to go any further") and then represented successive jumps 70 then 40, then 6 and finally 9 to give 125. When illustrating the solution to 289 add 17 he drew a line with the ends marked 289 and 315 and starting from 289 did a left to right jump of 10 and then 7. The reason why he put 315 at the right end and not a smaller number was "Just in case it went over".

Child 5.1 demonstrated an interesting variant on the use of the number line and on the use of partition to obtain a solution to 24 + 32, (Figure 6.4 refers).

> You can do like... number line. So you can put zero there (left end) and then put one hundred there (right end) ... and then we want twenty, thirty... The thirty... you have to do little bits here (in between 20 and 30) so you can do that four there and two there, and then we have to do jumps and add them altogether... put fifty-six there and see how many jumps it takes to go to fifty-six. Because we have to get to fifty-six. Twenty-four add thirty-two... two and thirty and twenty and four. What we do is two add four (jump from 2 to 4) equals six and twenty add thirty (jump from 20 to 30) equals fifty and then we add them together. Fifty and six, fifty-six. (Child 5.1)

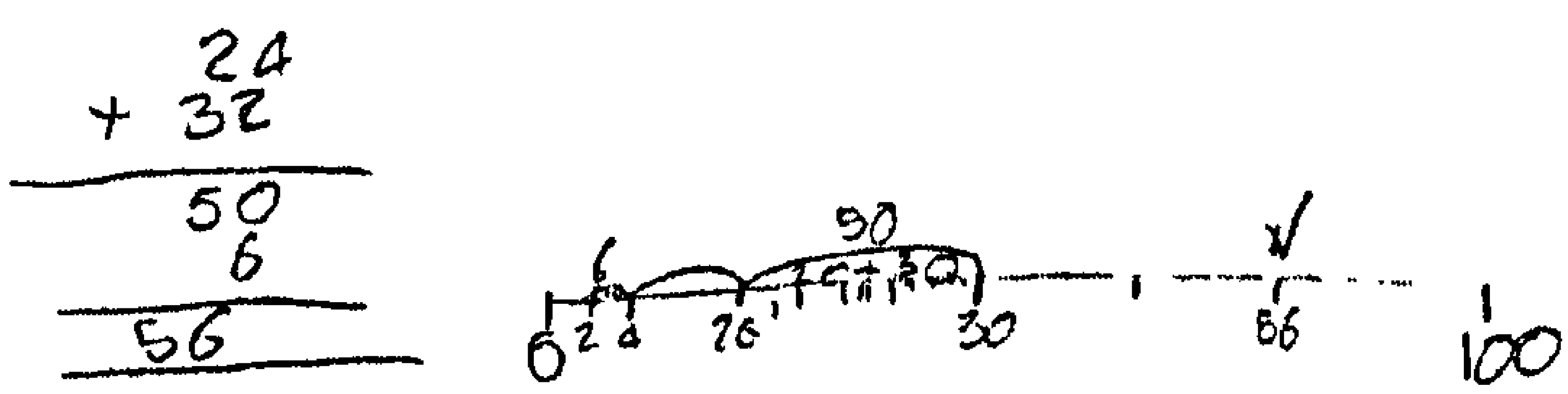


Figure 6.4: Mixing partition with number line method (Child 5.1)

From a perceptual perspective the child indicates that the line has numbers and little marks ("little bits") but apart from being the indicators of the position of a number, there appears to be no conceptual sense of why such mark may be there or of what the relationship between each may be. Note for example that there are twelve almost randomly placed 'bits' between 20 and 30 and that the interval between the 'bits' appears to have no relationship (note the interval 0 to 56 and the interval 56 to 100).

From an operational perspective, the partitioning process guides the jumps made on the number line, but these jumps have no relationship between what is happening and the eventual outcome of the process. Note how the sum of 2 + 4 and the sum of 20 + 30 is indicated as a jump *between* each of the two numbers and not a jump representing the addition of one number to another.

The interviewed children within Y5 tended not to use the number line as a representation of choice to deal with addition problems. In a sense, it seems to have served its purpose. As a tool it facilitated the development of partitioning within earlier years, but returning to its use seems to cause problems with the children who appear to see it as a line with a beginning and an end, but display little implicit awareness of its underlying characteristics nor can they remember how to use it operationally.

During the first observation of Y5, the class teacher used a number line described as:

> ... a nice number line. It's kind, because it's coloured in. In tens, so we've got red-blue-red-blue for each ten [this number line had the tens marked and labelled, while alternate intervals in between the tens were coloured red-blue]. (Y5 Teacher, Observation 1)

to reinforce the multiplication and division of numbers by 10 or 100. The children were reminded that multiplying by ten meant a shift of the numbers to the right, whilst dividing by ten meant a shift of the numbers to the left. Although it was not made explicit, the coloured number line was used as a representation to indicate this through the process of counting forwards or backwards in tens. The advantage of the colours appeared to be seen in the way they could support counting. To divide 70 by 10 the teacher explained:

> ... you have to count up how many tens in seventy? So we've got one... in jumps, here is seventy, so we've got [starts from right (70) towards left] one, two, three, four, five, six, seventy and that would start... seventy divided by ten. How many? [Teacher counts jumps

> 1, 2, 3, 4, 5, 6, 7]. Because seventy to sixty counts as one.
>
> (Y5 Teacher, Observation 1)

This use of the number line to illustrate division by ten was a feature of the interviews with the children following the lesson. Each child was presented with the actual example used by the teacher that is 70 x 10. None of the children was able to illustrate this satisfactorily using the number line. Child 5.3 indicated that it did not make sense with the number line, but neither did her alternative solution:

> [It is seven] because I took seven away from the ten and that equals seven. Because ten times seven equals seventy, but seven add three equals ten, so I just took three away from the seven, equals seven. (Child 5.3)

Child 5.5 recognised the general rule to find the answer to 70 ÷ 10 but illustrated confusion about the number line.

> Child 5.5: I move it one column to the right... cause dividing to get smaller.
>
> I: Did your teacher use anything?
>
> Child 5.5: Yeah! A number line, but I don't know that... I find it too easy.

When asked to attempt it with a number line, Child 5.5 wrote 7 and 70 at the ends of an empty line and 35 in the middle but could not solve the problem.

Child 5.1 recognised that the number line could help her count back. She added that her teacher had said, "If you get stuck on divide... if it is ten or one hundred get a number line and just help yourself". However, she illustrated that whilst she could divide by ten, multiplication by ten was completed the same way. When later asked to find the answer to 30 x 10 she drew a number line with the ends marked 0 and 30 and made jumps of ten from 30 back to 0 to give the answer 3.

Clearly, the evidence from the interviews with the children within Year 5 is demonstrating that they do not possess conceptual understanding of the way in which the number line may be used as a tool to support arithmetical operations. The stage in these operations have been partially remembered, place the big number on the right and the little number on the left but this not only applied to addition and subtraction but was now being confused with multiplication (Child 5.5) whilst a process that may support multiplication was confused with that used for division (Child 5.1). Misconceptions associated with use of the number line were becoming common.

6.2.6 The Number Line and Division: Year 6

One of the two lessons observed within Year 6 associated division with the use of the number line. However, in something of a similar fashion to the teachers within earlier years, the Year 6 teacher seemed to make as an assumption that the children within Year 6 shared her embodiment of the number line. She used it to represent 52÷10 and her final representation is indicated in Figure 6.5.

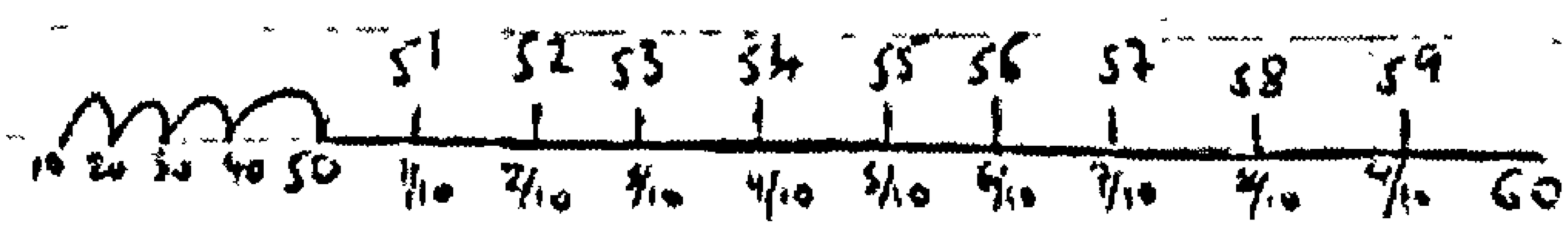


Figure 6.5: Diagrammatic representation of the use of the number line to solve 52÷10 (Y6 Teacher)

No line is associated with the jumps from 10 to 50 and zero is not pinpointed (and in her explanation below, the jumps start at ten) so the first jump 0 to 10 is neither apparent nor included (the answer as illustrated is 4 jumps of ten with a remainder of two). A line is completed from 50 to 60, but it is difficult to determine what the fractions are intended to represent since in her explanation the teacher indicated:

> How many groups of ten can I make out of fifty-two? What is my remainder? Five groups of ten. I've got five remainder two left over. We've got ten, twenty, thirty, forty, fifty [drawing jumps]. So five lots of ten make fifty. The next number the ten can divide exactly into would be sixty [construct a line from fifty to sixty and then divides it up to write fifty-one, fifty-two, fifty-three,...]. Between fifty and sixty there's ten numbers [writes the fractions 1/10, 2/10, 3/10,...] If I do fifty-two divided by ten, I have five lots of ten here [the jumps] and then I've split the remaining two numbers and divide them by ten as well. So I've divided fifty by ten and these two numbers here (50 and 60) I've divided them by ten as well. Fifty-two divided by ten is five and two tenths.
>
> (Y6 Teacher, Observation 1)

There was no explicit indication that it is the interval 50 to 60 that was being considered as a new unit that could be partitioned, but instead the teacher refers to the two numbers 50 and 60 that have been divided by ten.

After further examples, the children were required to practice the division of typical problems such as 28÷5, 24÷10 and 47÷10. During this session, none of the children used a number line, although when working with a group the teacher did so. None of the children interviewed immediately after the lesson volunteered information about the use of the number line during the lesson although they could give some indication of the substance of the lesson and in particular an associated activity that involved rolling a die:

> We had like twenty-two divided by ten and we had to put the answer in a fraction, say the remainder... ten times two is twenty. I know the remainder is gonna be two. I'll do two and then I'll do two tenths. (Child 6.3)

> We were dividing and we had some roller dice stuff... and then we'll be getting the numbers, then we had to roll the dice again to get a divider number, so if it's sixty-four divided by five, you'll get twelve remainder four, so then you'd write it (12r4)... so it will be four fifths (12 4/5). (Child 6.1)

> We found out about dividing. First we divided by five, then by ten and then we had two dice. We had to roll a dice and if it landed on a five and a six it would make fifty-six, so we have to roll another dice and whatever number that was, we had to divide it by. (Child 6.4)

> We rolled two dice and got a number. Say it was five and four, that would be add up to fifty-four. We had like seven divided by four and then see how many fours there are in seven, so there is one and then we'd have remainder three. (Child 6.5)

Each of these children articulated the content of the lesson in an episodic way with a specific example to expand their explanation. Only one child provided a response that might be determined in a more generic way, drawing upon wider knowledge than that given during the lesson but his explanation will be considered in the summing up after the next two sections on fractions and decimals (§6.4, Figure 6.14).

None of the children volunteered information about the use of the number line for aiding the solution of a division during the interviews. When Children 6.1 and 6.5 were each asked twice whether there was an alternative way to the one they had just written, they repetitively answered "No!" Child's 6.1 suggestion for an alternative way was:

> You could use a dice and then you could use your brain to help you... to work it out.
> (Child 6.1)

While when Child 6.5 was asked to say whether his teacher used something to help her solve a division sum, he responded:

> No. She did it that [without using the number line] way. (Child 6.5)

Child 6.2 on the other hand, followed a similar approach to the teacher, but without direct reference to the number line:

I:	Is there anything to help you? [For 64÷10]
Child 6.2:	Coz it's divided by ten, I know there is six tens in sixty and there would be four left over, so I know that would be the remainder.
I:	Why is the four left over, four tenths?
Child 6.2:	Because between sixty and seventy there is ten numbers...
I:	How do you know it's ten numbers?
Child 6.2:	Coz you got sixty-one, sixty-two, sixty-three, sixty-four, sixty-five, sixty-six, sixty-seven, sixty-eight, sixty-nine [counting on fingers]. That's how I know and that's why I put it there. So basically you take that away, coz you're dividing by ten, so that's sixty-four... then instead of putting it like a divise time, just put that there and put the ten there. So that's like divising like that. So that's the sum and the answer.
I:	Is there anything you can use to solve it easier?
Child 6.2:	No.

Child 6.3 first solved the division without using the number line and provided a correct solution to 53÷5. When asked whether there was anything she could use to solve the division, she explained:

> Draw the number line and put at the beginning fifty and... at the end sixty... I can draw little marks to say fifty-one, fifty-two, fifty-three, ..., fifty-nine and then I've got to the fifty-three, coz that's where I've got to end up. I've counted three from fifty... I know that I've got to have a remainder, because I can't do five times five again, coz it will make fifty-five... ten remainder three. (Child 6.3)

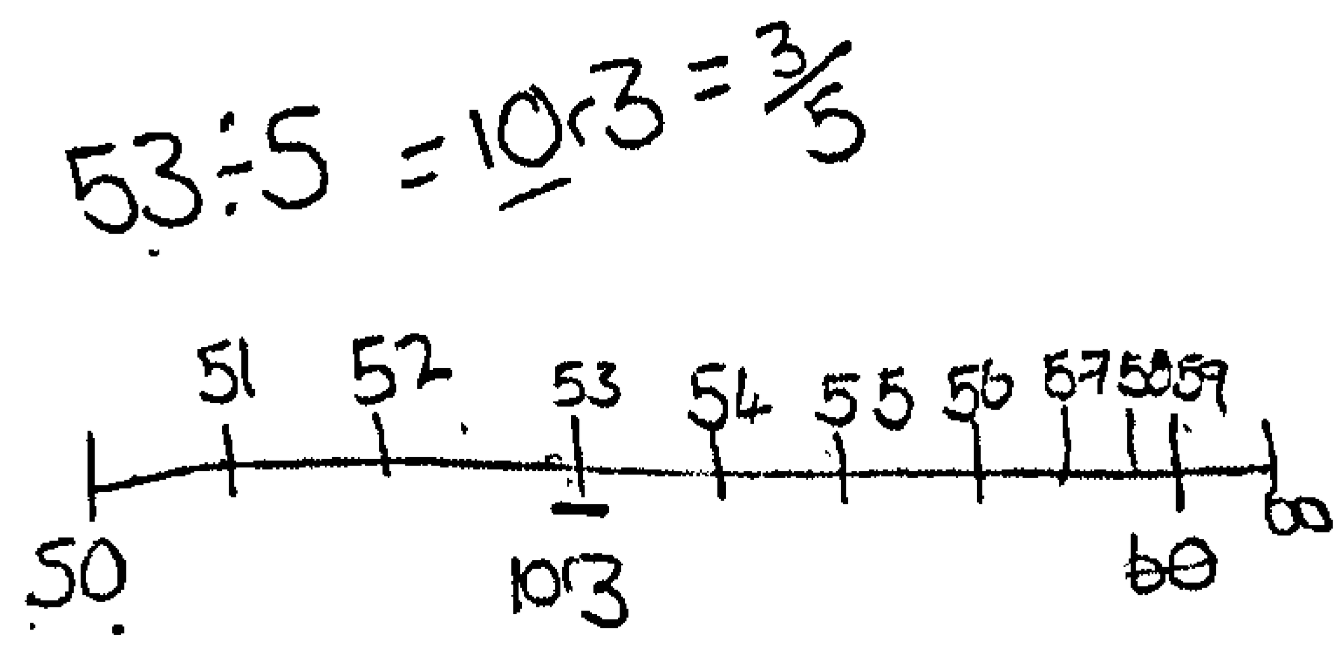


Figure 6.6: Division using the number line (Child 6.3)

Child 6.3 did not draw any jumps to get to 50; she knew the result of dividing 50 by 5 could be obtained from multiplying 5 by 10. Consequently, she only considered the segment 50 to 60. This she partitioned into units but did not identify fractional parts.

In contrast, Child 6.4 was asked to find the answer to 33÷5 by using the number line. He constructed a 0 to 33 line, provided an illustration of the jumps of 5 to 30 but did not subdivide the interval 30 to 33. He gave the following explanation as he completed the problem:

> A number line. Draw line, put thirty-three here [right end], put a nought [left end], jump five (from the left end), jump another five that will make ten, another five will make fifteen... [carries on making jumps] add five will make thirty. So it's six... to divide by thirty... and then you got to work the three out. Three fifths. Six and three fifths.
>
> (Child 6.4)

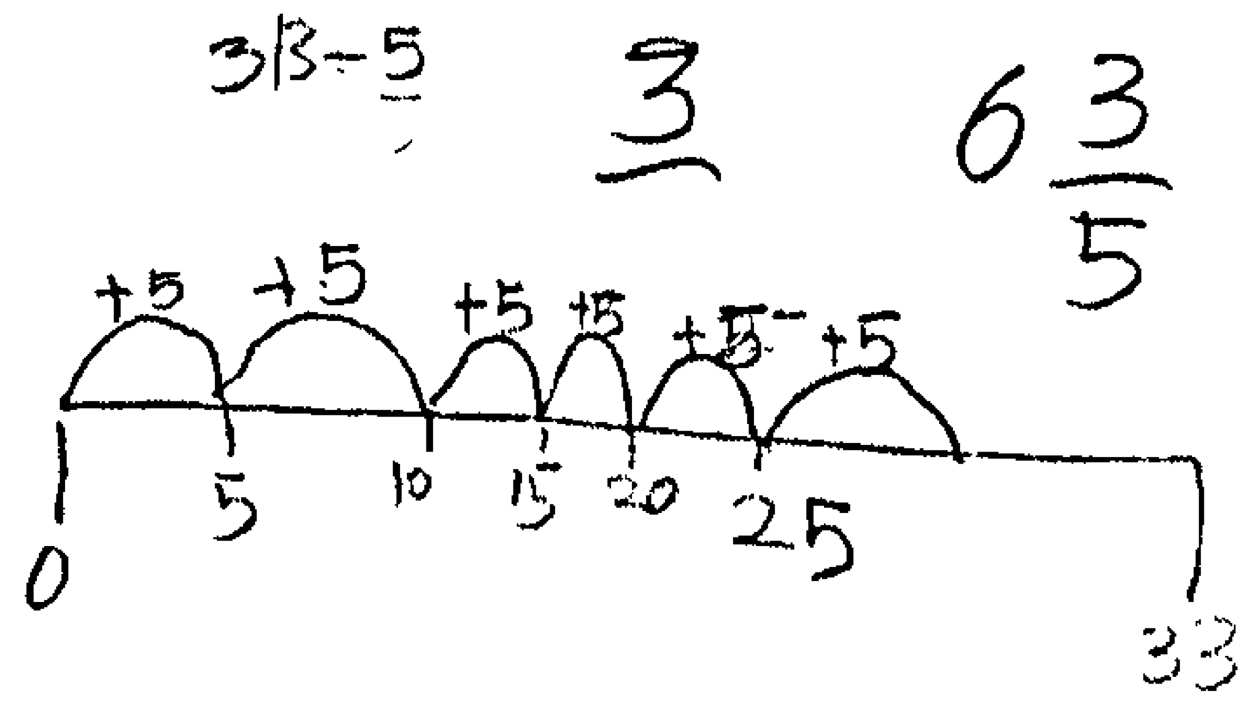


Figure 6.7: Division using the number line (Child 6.4)

It is interesting that the teacher's explanation and the children's use of the number line for division implied the use of repeated addition with a remainder left over expressed as a fraction. In this sense, the relationship between multiplication and division was being confirmed and as implied in the comments of Child 6.3, implied knowing multiplication tables means you can do division. It might be concluded that by the time these children were dealing with division in Year 6 the number line has served its purpose as a tool. They were able to solve the presented division problems and reflect on the relationship between a remainder and its symbol representation as a fraction of the divisor. From Sfard's (1991) perspective they had interiorised and condensed the operational thinking that was associated with coming to terms with the use of the number line in the process of division but in Dubinsky's (1991) terms the "de-encapsulation" that might be associated with unpacking the solutions to consider the actions and process in terms of the number line was a little insecure.

6.3 The Number Line and Developing Understanding in Fractions

Five of the seven lessons observed within Year 4 were associated with the concept of fraction. During these lessons, the class teacher made extensive use of several items that were associated with the notion of number line. Number line related tasks involved, for example, an unmarked stick approximately one metre long, empty number lines, a segmented line with the ends marked 0 and 1, a segmented line with the ends marked 0 and 5 and a variety of other lines with the left end marked 0 and the right marked with various numbers such as 60 or 100. Each of these representations were used at different stages within the development and were accompanied by other representations that included counters, shapes that required partitioning and shading, and verbal reference to cakes. This discussion focuses solely on the use of number line representations.

During observation 1 the teacher introduced the objective "to be able to find fractions" because this "will help you (the children) find the fractions of a whole".

It was obvious from the initial exchange that the children had experienced notions of half and quarter and the equivalence of two quarters as one-half. To develop the introduction the teacher used the unmarked stick:

T: If that's zero (left end) and that's one (right end) what number is in the middle?
C: A half
T: How do you know?
C: It's half-way between zero and one.
T: What if I wanted to find a quarter? It's half-way between. If that's a quarter, what would this in the middle be?

C: Two quarters.
T: So if that's the first quarter and that's the second quarter what would that one be?
C: Third quarter.
T: So a quarter, two quarters, three quarters, four quarters and four quarters is... a whole.

During this phase of the lesson, there was explicit reference that the stick was a whole. It was a stick with the ends marked. Though it carried the implication that it was a unit interval partitioned into four parts this was not made explicit, the focus being upon correctly naming and establishing equivalences between particular points on the stick. In this sense, the development carried the same features as lessons observed within Years 1 and 2, correctly naming and ordering points but the concern now was fractions and not whole numbers. The above sequence was then repeated to order eighths and indicate equivalences between eighths and quarters. The teacher concluded by saying:

> Whenever you have a fraction in between a number, it's always gonna be a half, because it's in the middle. (Y4 Teacher, Observation 1)

The phase continued with class activity for which the teacher distributed a sheet of paper with a line on it.

T: I've given everybody a blank what?

C: [A variety of responses were called out that included] Ruler! Paper! Line!

T: We're gonna call it a line. It's gonna be our number line. Put zero at this end (left) and one at this end (right). Where would you put a half?
Hands up if you put this (1/2) as well.
Did you also find that it was in the middle?
If I fold this piece of paper see if I have a half in the right place. Now find a quarter and three quarters.

The children's subsequent experience included marking the positions of 2/4, 5/10, 3/4, 10/10, 4/4 and 2/2. Subsequent lessons contained similar activities that ranged from reinforcement of fractions of an interval as above, to identifying fraction of magnitudes such as 2 1/2, 4 1/2 and 1/2 of a line segment marked 1 to 5.

During the interviews, it was not unusual for children to recall particular activities, but omit the drawing of a line segment although they focused on the numbers. For example Child 4.2:

Child 4.2: [The teacher] gave us a line like that [made a mark and wrote 1] and like that. To the right of the line I made a mark and wrote five and then two, two and a half, four and a half (See Figure 6.8).

I: Where did you have to do those?

Child 4.2: On a line on paper.

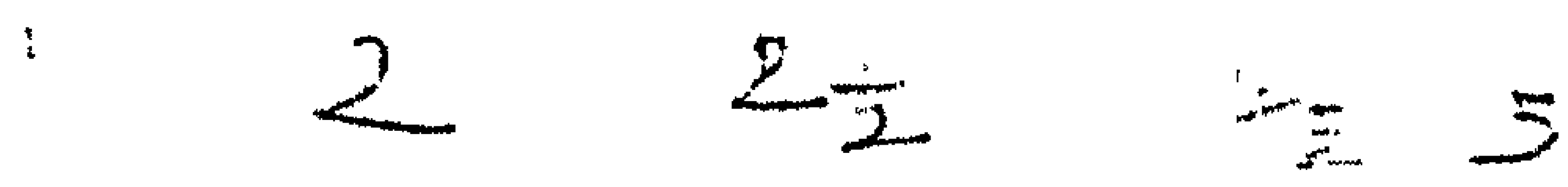


Figure 6.8: Writing numbers in an order without a line (Child 4.2)

Child 4.2 mentioned the word "line" twice, but did not pinpoint numbers by defining intervals and partitioning these. He simply ordered the numbers.

Activities associated with the development of the fraction concept frequently focus on the use of geometric shapes that can either be easily folded into halves and quarters or possess reflective symmetry. Within the Y4 sequence of lessons, folding a number line became a particular feature of the teacher's presentation. During observation 1, the teacher indicated:

> A way of checking the half is by folding the sheet in the middle.
>
> (Y4 Teacher, Observation 1)

When the children were asked to draw a number line with the ends marked 0 and 60 and then asked to find a half, with no reference to the fact that they were finding half of 60, she suggested that the children should check whether or not they have marked the correct point:

> ...fold this piece of paper, see if I have a half in the right place.
>
> (Y4 Teacher, Observation 1)

This approach was practiced by many children during the lesson, who folded their individual piece of paper and consequently received their teacher's praise:

> You've folded your paper. Well done! (Y4 Teacher, Observation 1)

During the interviews, one child clearly admitted doing so:

> I've got a way of checking the half! You could fold the sheet in half... or you can just go like that... go in the middle of there, there... and it will be there [placing thumb and index

> at either ends and drawing an imaginary line from the palm bone of the index finger]. (Child 4.3)

A focus on the middle of a magnitude dominated the teacher's instructional practice. After drawing a number line segment marked 0 and 10, 5/10 was marked in the centre of the segment and the teacher again explained that:

> A half of anything will be exactly in the middle. (Y4 Teacher, Observation 2)

This was again something that children clearly remembered:

> There's only one half in the middle... the proper half. (Child 4.3)
>
> Anything that's in the middle equals a half. (Child 4.4)
>
> Coz in the middle, everything is a half. (Child 4.2)

The children were presented with an activity from observation 1 during the interview that followed the lesson. The children were asked to mark the two ends of a line segment as 3 and 5 and find what is in the middle and then identify the positions of 3 1/4, 3 3/4, 4 3/4.

Only Child 4.1, marked the middle of the line as well as the halves and quarters correctly. Three of the children (4.2, 4.3 and 4.4) correctly marked 4 in the middle. But marking the fractions was problematic. Child 4.2 explained what he was doing whilst completing Figure 6.9

> Child 4.2: One and a half [wrote 1/2], one and three quarters [wrote 1/3]... one over four [wrote1/4], one over five [wrote 1/5]... one over nine, ten tenths, ten over eleven, ten over twelve, ten over thirteen, five.
>
> I: Why should ten over ten come after one over nine?
>
> Child 4.2: Coz nine comes before ten....

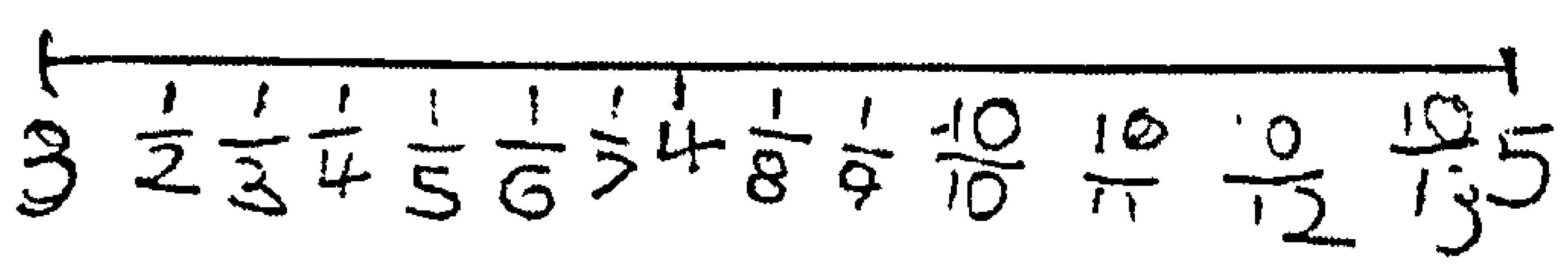


Figure 6.9: Applying whole number operations to fractions (Child 4.2)

It is noticeable even Child 4.2 had clearly marked 4 as the middle of the 3 to 5 line segment, this was ignored when the fractions were inserted. There was no attempt to pinpoint the position of these fractions. They were simply the child's perceptions of the way fractions were ordered but clearly the child's knowledge of fraction is being influenced by whole number considerations — the larger the number the larger the fraction. However, 10 seemed to be of particular significance. After the completion of 1/9, the child wrote 10/10 and, almost as if he had forgotten the sequence that he was writing, 10 now became the numerator although the denominator continued to reflect the order of natural numbers.

The influence of whole number arithmetic was also identified in the response of Child 4.3 who after marking the required whole numbers 3, 5 and then the middle, 4, proceeded to write under each whole number fractions formed from the sum of the numerator and the denominator that would equal the whole numbers (Figure 6.10) at the same time giving the following explanation:

Child 4.3.	... Four ones [writes 4/1, under the 5], three twos [writes 3/2, under the 5].
I:	How do you think about them?
Child 4.3	You just have to add every number up to five. If you had three add two is five. It's two threes... thirds it's five.

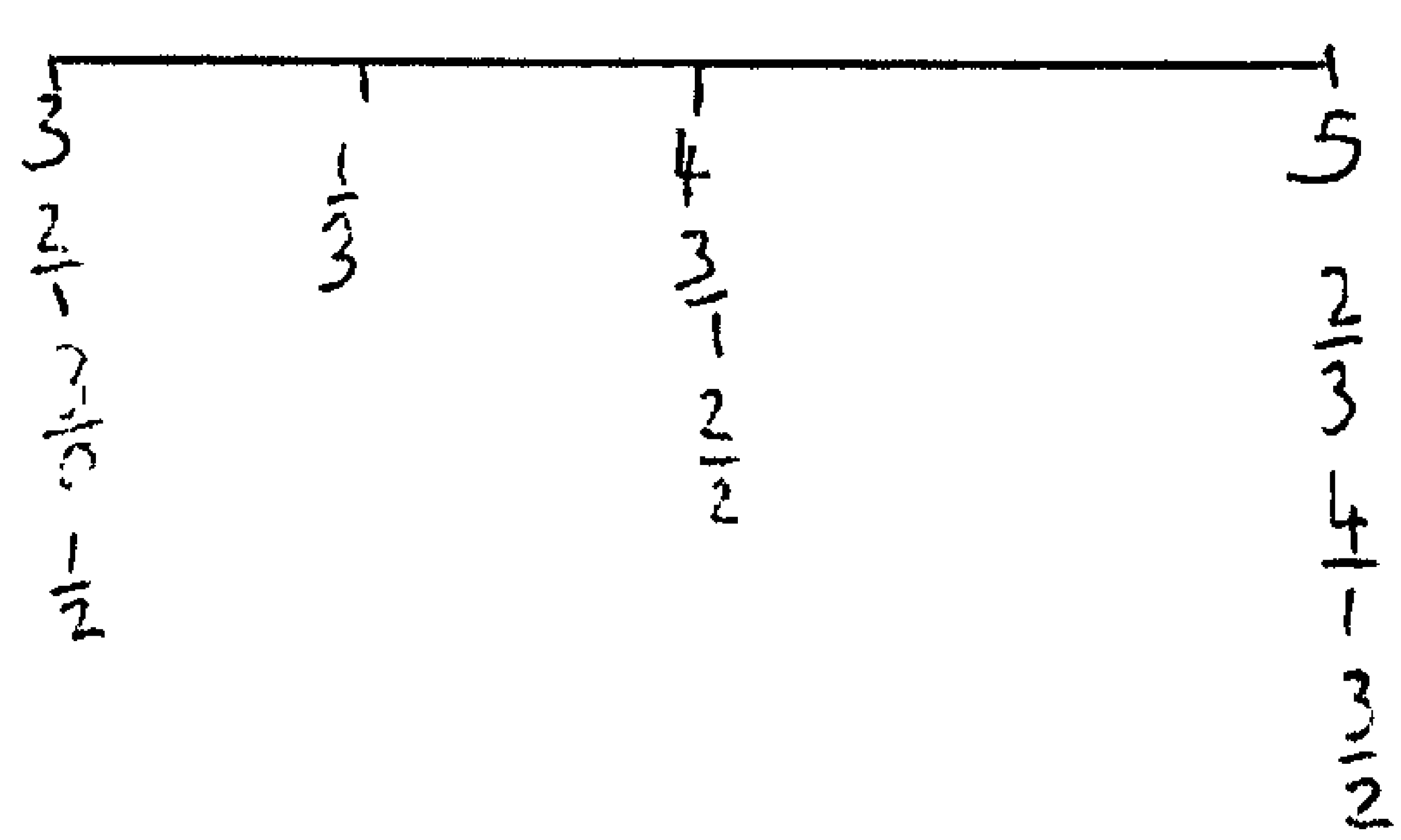


Figure 6.10: Fraction misconceptions and their relationship with whole number (Child 4.3)

However, the misconceptions of the children may have arisen from the way in which the material was presented to them. After drawing a number line from 0 to 10 on the board, the Y4 teacher marked 5/10 in the middle, and then, in the order 1/10, 9/10, 3/10, 4/10, 7/10, 1/2 placed these fractions at the places where numbers 1, 9, 3, etc. should have been, explaining and reiterating her earlier comments saying:

> I drew a number line zero to ten. Where would five tenths go? Draw it in the middle. Remember. A half of anything will be exactly in the middle.(Y4 Teacher, Observation 2)

Child 4.4 was given the same segmented line and asked to mark the fractions 5/10 9/10, 3/10 and 10/10. She repeated the teacher's action and marked 5/10 in the middle and the other fraction numbers at the approximate places after counting in small intervals from zero. The child also implied an equivalence relationship between number 10 and 10/10 by writing one under the other. When asked to pinpoint the numbers 5 and 3 and explain why (after already pinpointing 1/10, 9/10, 3/10), she counted in ones from zero:

Child 4.4: [marking 5] Coz you count from zero and then one, two, three, four, five and it equals there...

I: What about three?

Child 4.4: [After marking 3 explains] You write the three tenths, you just go one past.

I: What does the three-tenths mean?

Child 4.4: You got zero and one and two and then you put three tenths.

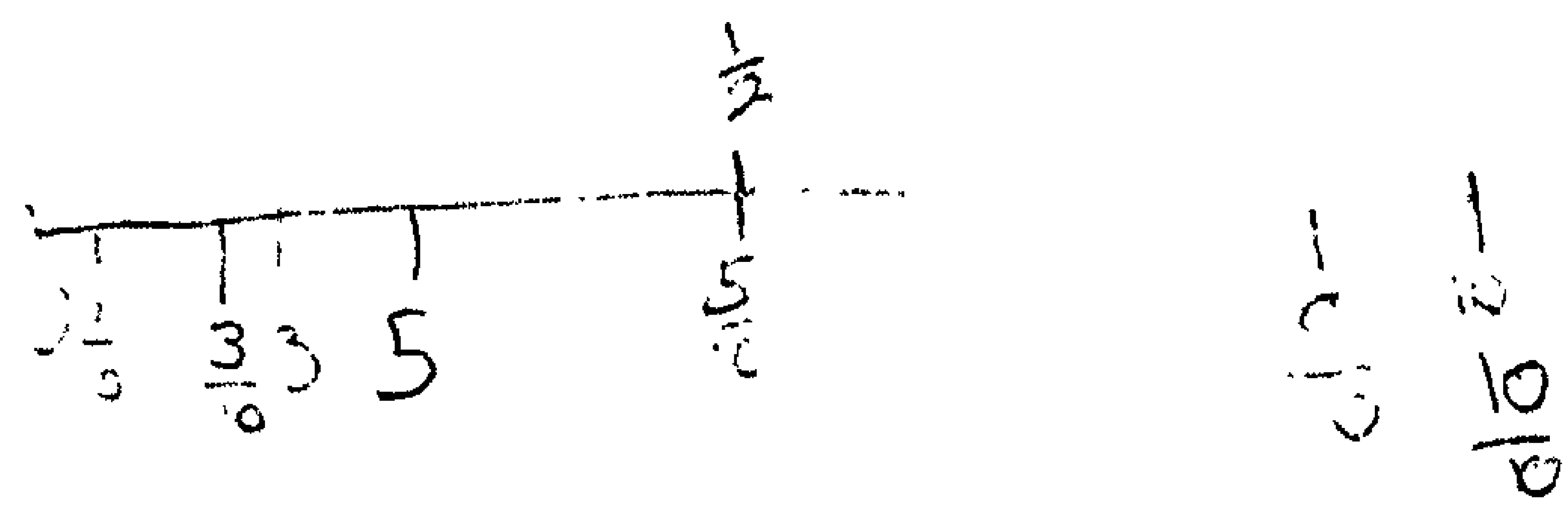


Figure 6.11: The relative positions of fractions and whole numbers (Child 4.4)

Representing fractions or whole numbers on this line, Child 4.4 used a counting approach. That 5/10 of the segment was 5, was not made explicit by the teacher and neither did it appear to be understood by the child.

Though the Y4 teacher extensively used the term 'number line' and used representations that may be associated with a number line, the essence that a fraction on a number line derived from the partitioning of a whole, whether that whole was the interval represented by one unit or the segment represented by multiple units, was not emphasised. Within the teaching, the number line was used as a mechanism to find fractions of large numbers, but notions such a "fold" to find the fraction of a magnitude carried the implication that it was the half of a surface that was being considered — the fold was not cross-referenced to the magnitude. The notion that a number line expresses magnitude through continuity was not made explicit in any of the observations within this class or any of the other class. A consequence was that the overall impression gained from the teaching observations was that the children were presented with the task of reconstructing their knowledge of the whole number system to include fractions of wholes but without conceptual links being made between the two. Again, the number line was used as perceptual support to order numbers and in the instances identified above, the ordering of whole numbers and fractions were seen to be discrete activities that appeared to have no relationship with each other.

Lessons directed towards the use of the number line to consider fractions were not observed in any other class.

6.4 The Number Line and Developing Understanding in Decimals

Within Year 6, one lesson was observed where the number line was used to develop understanding of decimal fractions. The focus of this lesson was rounding of answers after division (see also §6.2.6) and may have been considered as part of the section on division. However, it was thought more appropriate to place these considerations, since they were more strongly associated with decimal, in a separate section. However, included in the activity associated with decimals were references to negative numbers.

In a series of successive activities, the children were asked to:

- identify the positions of 0.5, 0.15, -1, 1.3 on a segmented number line marked with 0 in the middle and 1 towards the right hand side

- one at a time, mark the positions of 8, 10, 8.5, 8.25, 9, -9 on an empty line segment with 0 marked in the middle

- to mark the number halfway between the numbers 1.5 to 1.6 that identified the extremes of a number line segment and then to pinpoint the positions of 1.57, 1.56, 1.58, 1.59.

These tasks took place during the introductory phase of a lesson with the objective to estimate prior to calculating the answer of a division sum so that:

> [We can] check if our original answer is close to our estimate.
>
> (Y6 Teacher, Observation 2)

During the subsequent interviews, the selected children were given identical number lines to those they had dealt with during the lesson. Placing numbers left to right with little reference to the notion of repeating the interval was common to all of the children. Child 6.5 did this for both positive and negative numbers and it was interesting to note that whilst he provided markers for positive numbers, no markers were provided for negative ones. This child was given a line with zero marked in the middle and was asked if he could put more numbers on.

> One, two, three, four, five, six, seven, eight, nine. Minus ten, but that would be minus one. (Starts writing left to right the negative numbers) (Child 6.5)

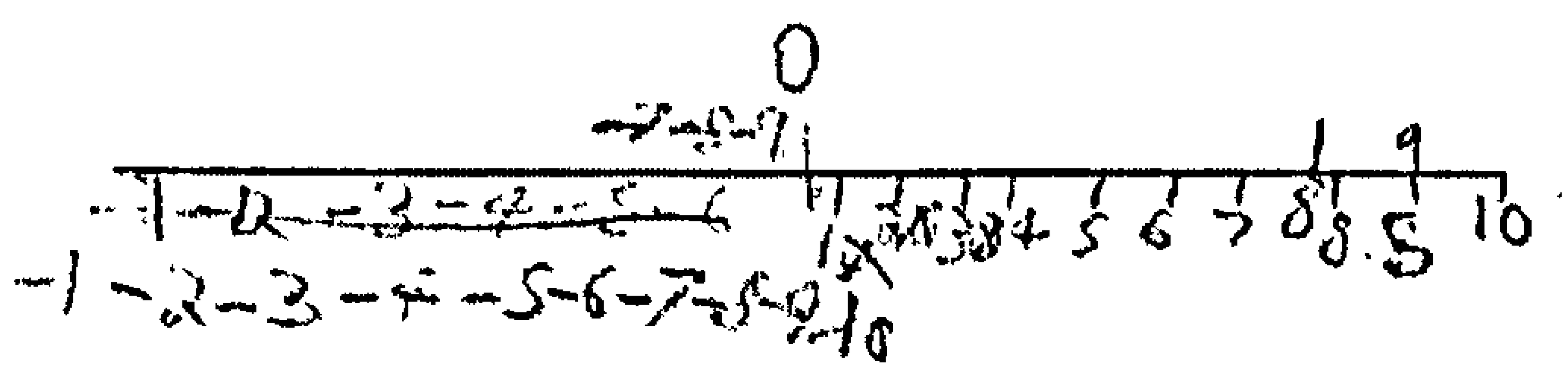


Figure 6.12: Counting left to right to put negative numbers (Child 6.5)

All the Year 6 children used verbal counting and made reference marks at some stage during number pinpointing:

> (On a line from 1.5 to 1.6) Halfway will be one point five five. You could put one point five six, one point five seven, one point five eight, one point five nine. (Child 6.3)

> (On a line with 0 in the middle and 1 to the right) You could put zero point one, zero point two, zero point three, zero point four,... zero point nine. (Child 6.1)

> (On a 0 to 1 number line) Naught point five that will be in the middle there. After the one you can put one point one, one point two, one point three. (Child 6.4)

On two of the lines, Children 6.1 (given a line with zero marked in the middle and one at the right end) and 6.5 (given a line with zero marked in the middle) provided verbal counting, but did not make marks, such as in Figures 6.12 above and Figure 6.13 below:

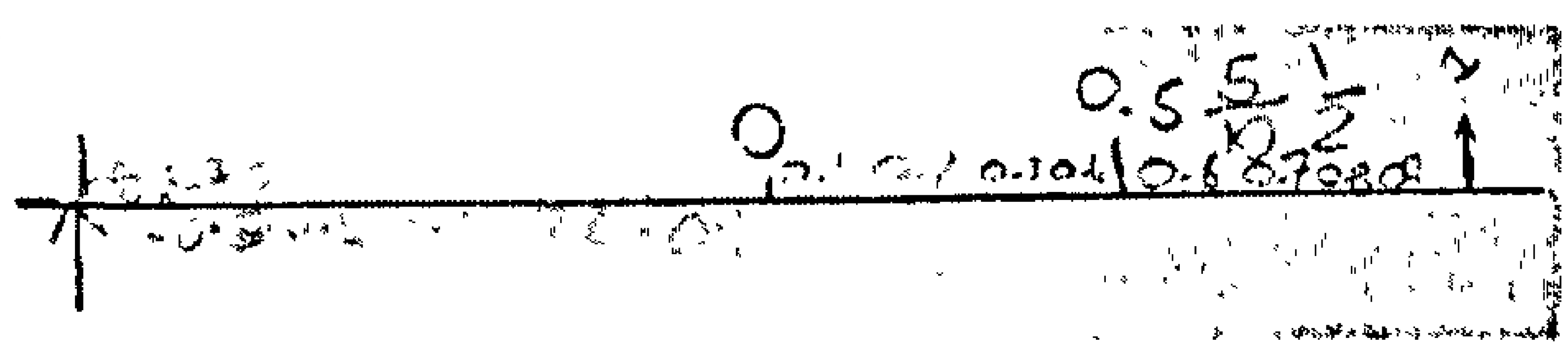


Figure 6.13: Counting left to right to put negative numbers (Child 6.1)

However, the responses of Children 6.1 and 6.5 were accompanied by comments on the lack of space on the number line, such as:

> ... there's no space left. (Child 6.5)

> You don't really get much room to spread them [numbers] out. Maybe I should've typed it [each number] in a bit more. (Child 6.1)

A considerable amount of space was taken up by actually writing the numbers, particularly those that were written first; with the result that there was no space or at least very little space for the final numbers. Both children wrote numbers both above and below a line in order to fit them all in the same interval (Figures 6.12 and 6.13).

When Child 6.5 was asked why minus seven was on top of the minus five (see Figure 6.12), he said:

> Child 6.5: Because there's no space left.
> I: What about the minus seven, the minus eight. Where would they go?
> Child 6.5: Trying to fit... next to there, or just move that down [crosses the number out and re-writes –1 to –10 left to right].

Although the extension or compression of numbers on the number line is possible, since this is an inherent property of the representation, Children 6.1 and 6.5 could not respond to it. Child 6.4 was asked to put negative eight point eight on a line that had zero marked in the middle and one at the right end. He said:

> It ain't enough paper. (Child 6.4)

With this statement, Child 6.4 turned down the options of extending or restructuring the number line. Only Child 6.2, when asked what he would put on a line segment, with zero marked in the middle and number one on the right end, provided an indication that he had conceptual understanding of the number line as a representation of the number system. After consecutively dividing intervals, the following exchange took place as he completed Figure 6.14:

Child 6.2: [marks 0.5, 0.25, 0.75] That would be minus one and I'd just do the same there. That would be minus naught point five... that'd be minus naught point twenty-five, minus naught point seventy-five...

I: Could you put any other numbers on this line?

Child 6.2: I could, but that would go into eighths and things like that. Between that (0) and that (-1) I'd do it into tens. That's one, two, three, four, five, six, seven, eight, nine (marking the notches for the negative decimal numbers). Minus zero point one, minus zero point two and so on.

I: Could you put a quarter on this line?

Child 6.2: Yeah! That's the quarter there (points at 0.25), that's the half (points at 0.5) and that's three quarters (points at 0.75).

I: Could you put minus 0.28?

Child 6.2: You could, but it would go into really silly fractions.

I: What do you mean silly fractions?

Child 6.2: You'd have to put it into... you'd have to find something that goes into twenty-eight, unless you get... split twenty-eight up and put twenty-eight lines.

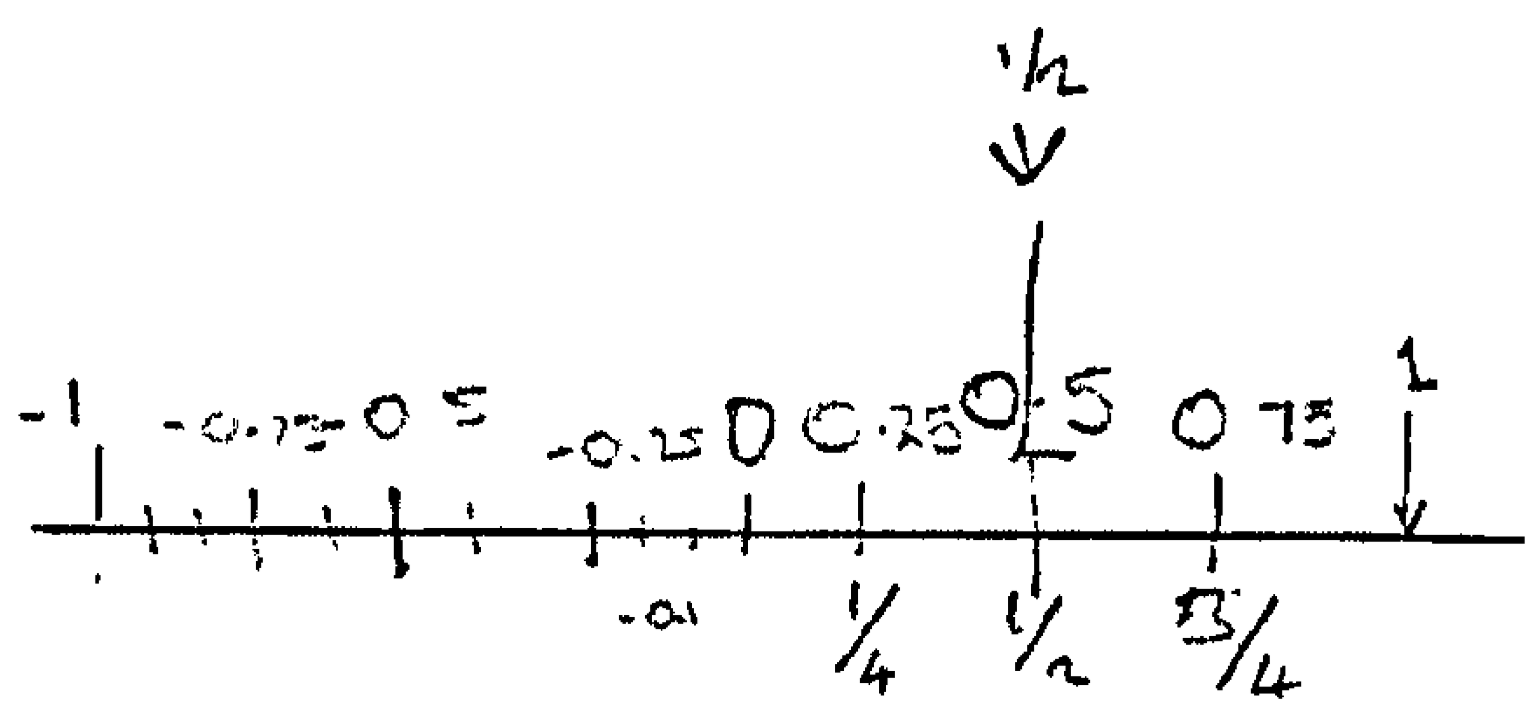


Figure 6.14: Generic response towards number line structure (Child 6.2)

Child 6.2 was the only child to begin to identify the relationship between the different forms of numbers, whole number, fractions, decimals and negative numbers and use this relationship to give a sense of his understanding of the number line. Neither during teachers' presentation during lessons nor during interviews with the other children was the potential richness expressed by this child apparent. It was a richness expressed during other aspects of the interview process. For example, Child 6.2 drew upon a specific example to focus upon the different solutions that may apply to a division problem within which there is implicit understanding that the number line segment between 60 and 70 becomes a whole that can be partitioned:

> We had a number, say sixty-four, we divided it by ten and then we had to show a remainder... and the remainder you put at the top and then you put the thing you divide it by down the bottom, so that will be four tenths. The amount of tens that go into sixty-four, that'll be six, so it will be six and four tenths [wrote down 64÷10=6.4=6r4=6 4/10] I've just put a decimal point. And then... that's more or less the same as that... as remainder... so I put an 'r' there instead of that (the dot) (I: Why is the four left over, four tenths?) Because between sixty and seventy there is ten numbers... Coz you got sixty-one, sixty-two, ... , sixty-nine (counting on fingers). (Child 6.2)

6.5 Chapter Summary

Herbst (1997) suggested that the possession of conceptual understanding of the number line permits the use of that line for the performance of arithmetic operations. Within this chapter the evidence from a series of classroom observations and a series of interviews with children who were present within the lessons, suggests that there was no explicit reference from teachers as to the conceptual foundation of the number line and consequently no explicit reference to it from children, although the comments from Child 6.2 suggested that for him this foundation has been laid.

Tracing through the observations and the interviews, we have seen that it is the number line as a tool that is used in an illustrative way to introduce conceptions of order, addition, subtraction, multiplication and division. As a tool, the number line is associated with indications of where the numbers to be added to or subtracted from should be placed to facilitate the action of completing a prerequisite series of jumps. Within the later years, we see that it is these jumps alone that become the major focus of attention. Children have the choice as to whether they actually construct a number line (Year 4 teacher) or it is assumed that they possess an embodiment of a number line that is similar to the teachers' (Years 5 and 6). Such an embodiment is always presented with a left to right orientation, is largely composed of whole numbers but does not directly bear reference to a repeated interval so that jumps may have some relationship to the scale of the number line.

Gradually the complexity of ideas presented to the children, immersed as they are in ambiguity, leads to the children's conclusion that arithmetical operations can be done in an easier way than that which encourages use of the number line. There appear to be so many things to remember; largest first, count-on in addition, largest first count-back in subtraction, re-structure knowledge so that subtraction becomes smallest first, count-up for subtraction, bridge through tens, count-on in tens, etc.

Although the number line or one of the various representations described as a number line, would appear to be used successfully in the development of partitioning, this hardly leads to conceptual understanding. The number line is not presented and developed as an abstract conception of the number system, but as a concrete model that supports actions. It is almost as if it is seen as an unsophisticated computer that will 'help' children solve arithmetical calculations or find fractions. But, as for most of us with more sophisticated computers, the children would not seem to possess the underlying understanding that would indicate how their computer is constructed. The emphasis placed on ordering numbers leaves the impression that this is what they have — a line with marks and numbers upon which they can do jumps and the size of these jumps is frequently constructed after counting. Even when we see the lessons on fractions we are left with the impression that it is not a number line segment that is being folded but a piece of paper with numbers on it.

Whilst the notion 'number line' remains part of the language used by teachers, a perceptual reference becomes increasingly less common. There seems to be an implicit assumption that the children have a mental embodiment of a number line which matches the teacher's embodiment so that during teaching there is an increasing tendency to simply provide visual references to jumps without associated reference to a number line segment.

Herbst (1997) indicated that the number line can be used as a pedagogical tool and enable the performance of arithmetic operations, as long as its feature of having a one-to-one correspondence between numerical statements and number-line figures are knowable. This suggests that to perform arithmetical operations with understanding, using a number line, an individual should possess the conceptual understanding that:

> one marks a point 0 and chooses a segment u as a unit. The segment is translated consecutively from 0. To each point of division one matches sequentially a natural number. (Herbst, 1997; p. 36)

Within Year 6, we see indications that children do not possess such understanding, since some children (6.5, 6.1) wrote numbers on a number line in such a way that one point on the line represented more than one number. This feature could be interpreted as a misunderstanding of the relationship between fractions and whole numbers and if this is the case, it is conjectured that the underlying continuity inherent within the number line has not been established for these children. They see whole numbers and fractions as two discrete systems.

The number line is a representation of the number system that may be used as an instructional material, but as we saw within Chapter 2 (§2.4.4, §2.4.5) there can be problems associated with the use of any representation as instructional material. Foster (2001) for example indicated that representations may be used in demonstration or developmental mode. There was no evidence from any teaching observation of the latter. Instructional use of the number line was associated with a demonstration by the teacher to be followed by practice by the children. Additionally, Foster suggested that a representation may "get in the way" of the learning of mathematics so that it may create obstacles. It is suggested that this is what is happening when the use of the number line is extended into the area of fractions and decimals in the instances considered within this chapter.

The ambiguity with which teachers talk about the number line, the number track and the hundred square does not support the notion of continuity that is unique in the number line. Therefore, it is conjectured that the knowledge reconstruction that children require to move from whole number to fractions is not supported through use of the number line, if their perception of that line is simply a mechanism for ordering whole numbers. At the end of this, it is the questions that were not asked that seem to take on more importance than those that were asked and one important question that should have been asked more widely is "Why are there gaps between the numbers on a number line?" Child 3.2 suggested that there was nothing in the gaps between the numbers.

> Those lines [above each number] are so that numbers don't touch... people might think it's all one number [if there were no lines]. (Child 3.2)

It is a question that would have added further insight to children's perceptions of the number line that is the focus of the next chapter.

Chapter 7: Children's Understanding of the Number line

7.1 Introduction

Within Chapter 6, we saw that the number line was essentially used as a tool to support children's notions of number order, the development of counting and as a support in the development of procedures to carry out arithmetical operations. As such a support, it can be seen that notions such as bridging tens or counting in multiples of ten were associated with backward or forward movements represented by jumps on the number line.

Number line and number track representations used within each class were oriented left to right and there was no evidence of an alternative orientation and some considerable ambiguity in the use of the word 'number line' to also refer to a number track, with the added suggestion that the hundred square was like a number line (§6.2.2). There was no evidence indicating conceptual differences between the number track and the number line, although the perceptual difference that the former started with zero and the latter started with one was emphasised by the Year 3 teacher (§6.2.3). The children's limited perceptions of the differences that emerged from this ambiguity frequently seemed to cause some problems when they attempted to consider fractions and decimals. In evidence of their perceptions of the intervals between repeated units, focussed upon spaces as means of separating numbers, something that was clearly identified on the number track, rather than an integral part of the number line which carried additional meaning.

Though children were provided with activities associated with estimating points on the line, carrying out these estimations was initiated through counting from zero (§6.2.1) but only within Y3 (§6.2.3) was there any indication that such estimations may be established through a more sophisticated strategic approach although even this was limited to the number line segment 0 to 10.

This Chapter focuses upon the meaning and understanding that children give to the number line as a representation per se, and in its analysis, the Chapter has three main components:

1. An analysis of questionnaires, distributed to children within 4 classes across the age range Y3 to Y6 within the school to establish their embodiment of the number line and the way they may label an empty number line. Y2 children were not required to complete this component as it was thought that they would have difficulty with any written components. Where appropriate the quantitative analysis is supported by evidence from written statements.

2. Focussed interviews with a selection of children from these classes, to establish in detail their understanding of what a number line is.

3. The accuracy with which children can estimate magnitudes on the number line. This phase also included the children from Y2.

§7.2 draws upon quantitative data to consider the first of the three issues identified above, that of the children's embodiment of the number line, through the analysis of the questionnaires presented to children within the classes Y6 to Y3. 74 children responded to this questionnaire, although the total responses for each item considered varied marginally because no responses were disregarded. §7.2.1 considers the children's embodiment of the number line as determined by their selected preference of a series of presented lines in different orientations whilst §7.2.2 considers the outcomes of opportunities given to them to place numbers on a series of number lines.

The second component, §7.3 draws upon qualitative data to obtain verbal indications of the way in which children perceive the number line, whilst §7.4 and §7.5 consider children's responses to the open estimation and then focused (through the use of an arrow) estimations of magnitudes on a 0 to100 number line segment. The raw data obtained from the questionnaire that was circulated to the children is presented within Appendix VI (Q.9) and Appendix VII (Qs.1-8). The Chapter ends with a Chapter Conclusion (§7.6)

7.2 Children's Embodiment of the Number Line

Within §2.2.5 we saw how Maturana & Varela's (1998) theory of embodiment was seen to be identified with the human perception of the world whilst that of Lakoff & Johnson (1980, 1999) attempted to establish a general description of human understanding concentrating on embodiment in the context of conceptual structures. Inherent within Maturana and Varela's theory is the inclusion of language, meaning and conceptual thinking whilst an important feature to emerge from Lakoff & Johnson's theory is that of image schema, as embodied concepts resulting from the mental-images learned through our bodily interaction with the world.

To gain a sense of the children's embodiment of the number line they were not asked to describe what a number line might look like but instead, through a questionnaire they were asked to respond to three issues:

1. Consider a series of differently orientated lines and to tick the one they liked best.
2. To place the symbols 0 and 20 on an empty number line.
3. To place numerals on a calibrated but non-marked line.

7.2.1 Embodiment Associated with Number Line Orientation

The children were presented with the following diagram (Figure 7.1) and asked to tick the line they liked most and explain why.

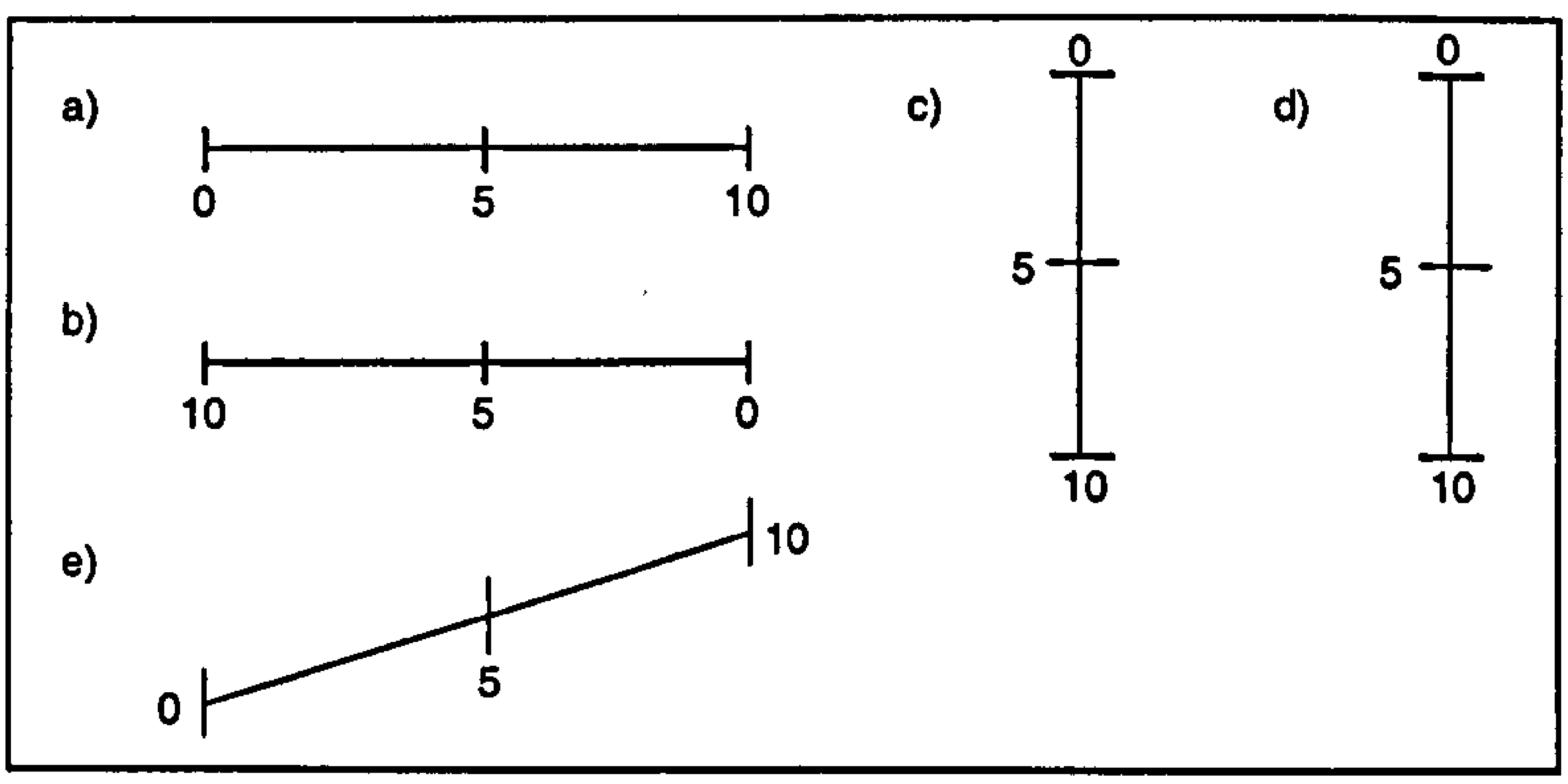


Figure 7.1: Tick the line you like the most. Explain why

Table 7.1 indicates the children's preferred number line orientation as identified from their responses to the above question.

Year Group	Proportion of line preference (%)					Class Total N
	Line					
	a	b	c	d	e	
Y6	80.0	10.0	0.0	10.0	0.0	20
Y5	41.2	5.9	11.8	5.9	35.3	17
Y4	73.7	5.3	5.3	0.0	15.8	19
Y3	44.4	0.0	11.1	5.6	38.9	18
Total	60.8	5.4	6.8	5.4	21.6	74

Table 7.1: Children's preferred number line orientation

It can be seen from Table 7.1 that:

- almost two thirds (60.8%) of the children indicate that they liked the horizontally orientated number line 'a' most, with the second most liked line being 'e' (21.6%).
- line 'a' is the most liked line within each year group but within Years 5 and 3 line 'e' is almost equally liked.
- lines 'b', 'c' and 'd' are liked by a maximum of two children within each class.

Consequently, lines which have a left to right orientation and are horizontal or slightly oblique are liked best by over 80% of the sample. Vertical lines or lines with right to left orientation are only favoured by the few.

The children's explanations for giving their preferences were distributed across a variety of reasons (see Table 7.2).

Year Group	Explanation for number line choice				
	Descriptive	How they felt	Comparison	Action related	Other
Y6 (N=20)	50	35	5	5	5
Y5 (N=17)	41	35	12	6	6
Y4 (N=19)	42	11	21	5	21
Y3 (N=18)	72	11	0	0	17
Total (N=74)	51.25	23	9.5	4	12.25

Table 7.2: Children's explanations for their number line preference

It was perhaps not unsurprising that the sense of the written responses bore a relationship to the classes that the children came from. Throughout all of the year groups, a focus upon one or more of the perceived properties of a chosen line was the most cited reason for a preference. Approximately 50% of the total sample, almost 75% of Y3, and slightly lower proportions within the other year groups (42% in Year 4, 41% in Year 5 and 50% in Year 6) associated their preference with easily identified properties through which the lines could be described:

> I like it because it is big. (Child 1(a), Y3)[8]
>
> Because it is straight. (Children 6(a), 13(d), Y3)
>
> Because it is tilted left and a nice shape. (Children 2(e), 3(e), 9(e), Y3)
>
> I like it because it is going up in five. (Children 5(a), 10(a), 11(a), 12(e), Y3)
>
> Because it is one of the biggest so you have more room. (Child 4(e), Y6)
>
> Because zero is first. (Child 10(a), Y5)
>
> I like this one because it is very straight and has the five dots in the middle. (Child 19(a), Y4)

Within Year 4 and Year 6 children, there were 1 and 5 children respectively, who mentioned the word "order" within their justification for preferring line (a):

> Because it is in order and right and because it shows. (Child 11(a), Y4)
>
> Because this one goes in order. (Child 6(a), 7(a), 9(a), 15(a), 16(a), Y6)

Interestingly, almost a quarter of the sample, biased towards Years 5 and 6 gave a reason associated with whether or not they understood one line better than other lines and thus, focused upon how they felt about a line:

> Because it's easier to understand. (Children 3(a), 10(a), 11(a), 14(a), 17(a), 19(a), Y6)
>
> Cause it's a normal number line and help you better. (Child 2(a), Y5)

---

[8] Child 1(a), Y3 indicates the way the written comments of individual children are referenced in this section. Each child has a numerical code whilst (a) indicates their chosen line. Y3 represents the class.

> Because I like it. (Child 18(a), Y3)

The second most popular explanation of Year 4 children liking a line was by giving reasons based on the comparison between lines:

> I like it because the other ones are backwards. (Children 17(a), 18(a), Y4)
>
> Because the other ones are more complicated. (Child 15(e), Y4)

One child from each of Years 4, 5 and 6 indicated that the reason behind their preference for a particular line was associated with an action-procedure:

> Because you can count up not down. (Child 13(a), Y6)
>
> Because I like to read from left to right. (Child 13(a), Y5)
>
> I like this one because it counts up in fives and I can count up in fives. (Child 4(e), Y4)

None of the Year 3 children gave responses that involved comparison between lines or that were associated with actions. However, one eighth of the overall responses involved either no comment or it was impossible, particularly within Years 3 and 4, to make sense of what was written down because of the misspelling of the words.

The children's over-riding preference for a number line with marked ends and with a left to right orientation was confirmed when they responded to the second issue — that of marking positions for 0 and 20 on an empty number line.

7.2.2 Embodiment Associated with Marking a Number Line

Three questions sought to gain a sense of where children may position just two numbers on a number line segment and how they would mark a calibrated but empty number line and a calibrated and labelled line. The first question asked the children to mark 0 and 20 on an empty number line (§7.2.2.1). The second invited them to place numbers on a calibrated line (§7.2.2.2). The third question asked the children to place more numbers on an already numbered line (§7.2.2.3).

7.2.2.1 Marking 0 and 20 on an empty number line

Table 7.3 indicates the way that 72 children from the classes Y3 to Y6 responded to the invitation to put 0 and 20 on the unmarked line.

Year	Ends			Other than Ends	Total
	Ends	Ends & naturals in between	Ends, middle, quarters	0 left end, 20 at quarter/middle	
Y6 (N=19)	85	0	5	10	100
Y5 (N=16)	47	47	0	6	100
Y4 (N=19)	79	16 (5% no marks)	0	5	100
Y3 (N=18)	94	0	0	6	100
Total % (N=72)	77	14	2	7	100

Table 7.3: Children placing 0 and 20 on an empty line (%)

From Table 7.3 it can be seen that:

- Every child marked the extreme left of the line with 0 and only 7% marked 20 at a point other than the extreme right.
- 77% of the children appeared to embody the whole of the line as a 0 to 20 line by marking the extreme points. A further 14%, drawn largely from Y5, although not asked to do so, also marked the naturals in between 0 and 20.
- 7% of the full sample, representing 1 child from each of Years 4, 5 and 6, appeared to see the empty line as portion of a larger line since after marking 0 at the left extreme of the line; they then positioned 20 at approximately either a quarter or half way along the line. Only Child 2 from within Year 6, selected for the sample to be later interviewed, divided the 0 to 20 interval into four sub-intervals by marking three notches to subdivide the line into four intervals.

7.2.2.2 Marking a calibrated line

Figure 7.2: Calibrated empty number line

The 74 children within Years 3 to 6 were presented with the number line as seen in Figure 7.2 and invited to consider whether they could put numbers on it.

Numbers were placed on the number line by 97% (70 children). 2 of the Year 5 children did not respond to the invitation. Of those that did respond, again 97% made use of whole numbers only. Of the remainder, 1 from Year 4 marked each point with the negative numbers from -6 to -1, whilst 1 from Year 6 (Child 2), responded by writing the decimal sequence 0, 0.2, 0.4, 0.6, 0.8, 1.

Those that responded by writing whole numbers did so in a way that enabled three categories to be identified: sequencing in ones from a nominated start, sequencing in 2's, 3's, 5's and 10's from a nominated start and the random placement of numbers.

1. Sequencing in ones.

- Number sequence starts at 0 and continues in ones to 5.

  Over 55% of those who responded by writing whole numbers wrote a sequence of numbers as in Figure 7.3.

  Between 58% and 69% of the children within Years 4, 5 and 6 provided this form of response but only 17% of Year 3.

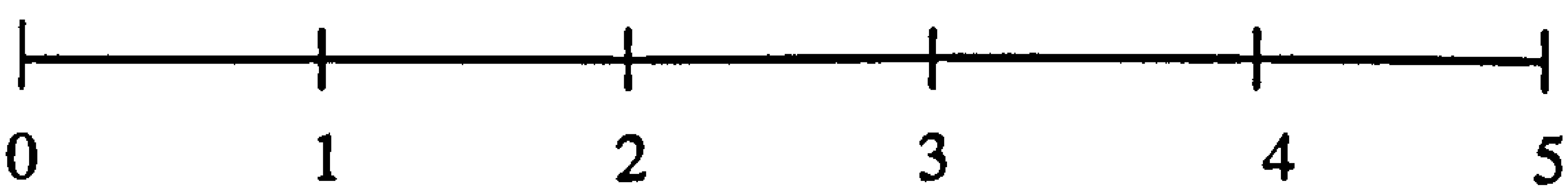


Figure 7.3: Labelling a calibrated number line (1)

- Number sequence starts at 1 and continues in ones to 6

  7% of the children (2 from Y4 and 3 from Y6) labelled the line as in Figure 7.4. No children within Years 3 and 5 provided this form of response.

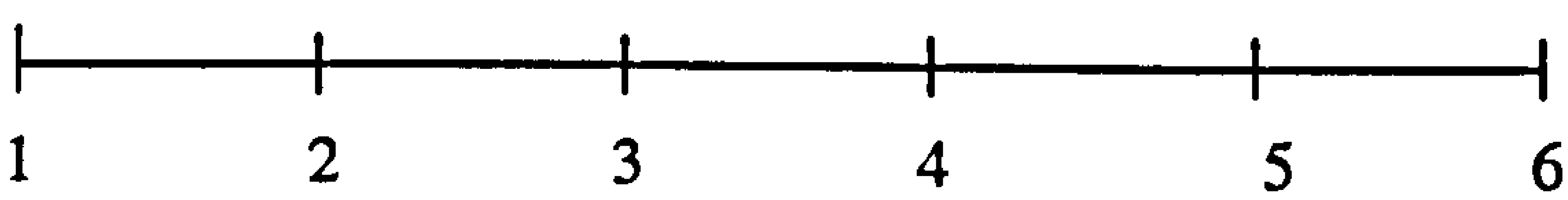


Figure 7.4: Labelling a calibrated number line (2)

Given that the question implied that the children had the freedom to label as they wished, not too much may be interpreted from the difference between Figure 7.3 and Figure 7.4.

- Alternative starting number and sequencing in ones.

  4 children (6% of those who used whole numbers), one from each of Years 4 and 5 and 2 from Year 6, labelled the calibrations by sequencing in ones from a chosen start. As in Figure 7.5:

15	16	17	18	19	20 (Year 6)
7	8	9	10	11	12 (Year 6)
5	6	7	8	9	10 (Year 5)
6	7	8	9	10	11 (Year 4)

Figure 7.5: Labelling a calibrated number line (3)

2. Sequencing with multiples of a particular number.

Just over one quarter of the children who used whole numbers (26%) to label the calibrations, used sequences based upon multiples of 2, 3, 5 and 10. This form of number labelling was most prevalent amongst the children of Year 3 where 66% (12/18) of the class correctly labelled using such a sequence. Just over 40% used multiples of 2, the remainder being almost equally divided between multiples of 3 and multiples of 10. One half of these sequences started with 0 whilst all but one of the remaining half started with the first multiple, for example, 2, 3 or 10. The remaining child used a sequence of tens starting from 20. Two children within Years 4 and 5 used sequences; one within each class using 5's the other 10's and one within each class also starting with 0. Only one chid within Year 6 used a sequence of fives starting at 5.

3. Non-sequenced placement of numbers.

One child from each of the year groups placed numbers with no discernible sequence on the calibration marks to provide markings such as 1, 2, 90, 1000, 10000, 99 (Year 1 child) and 0, 1, 2, 3, 4, 20 (Year 4 child). Two of these children appeared to provide a descending left to right sequence but wrote 10, 8, 9, 6, 5, 4. Two additional children from within Year 3 wrote non-sequenced numbers within the intervals of the number line as seen in Figure 7.6.

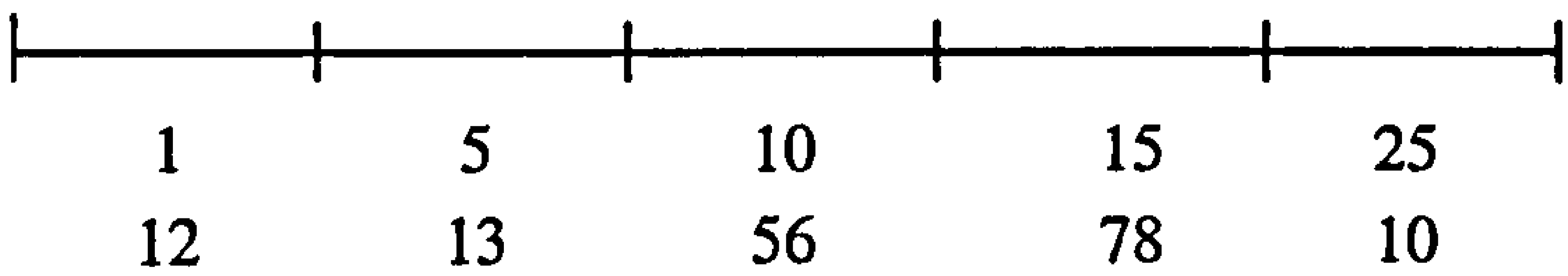


Figure 7.6: Labelling a calibrated number line (4)

In summary, it has been seen that in responding to the invitation to place numbers on the calibrated line almost 95% of the children used whole numbers and of these, two thirds sequenced in ones. Almost one quarter, largely children from Year 3, used some other form of sequencing and almost one tenth used some form of non-sequencing. Almost 70% of these children identified the left hand calibration as zero. Only two children within the whole sample successfully labelled with something other than natural numbers, one using negatives and one using decimals. It may be determined from these results that the greater number of these children embody the notion of numbers on a number line as whole numbers that are sequenced in one from left to right starting from 0. Year 3 children provided a general exception to the notion that the numbers were sequenced in ones since they extensively sequenced in multiples of 3, 5 or 10.

7.2.2.3 Adding additional numbers to a 0 to 5 number line

The limited evidence that the children's embodiment of a number line did not appear to include numbers as decimals or fractions, prompted the inclusion of an additional question inviting children to consider an already marked line and to indicate whether or not they could add additional numbers to it.

The children were presented with the number line, as seen in Figure 7.7, and asked whether they could place more numbers on it.

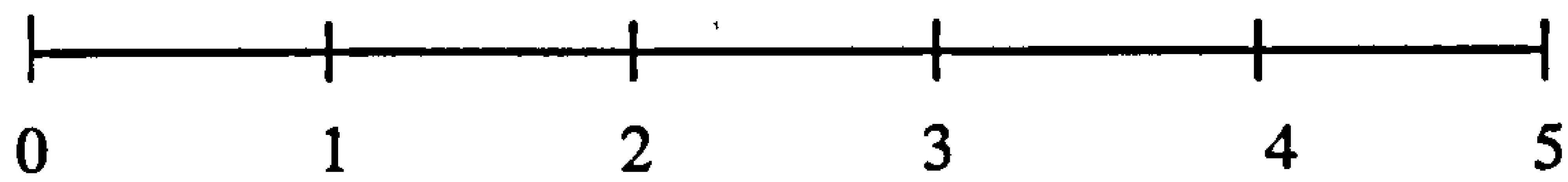


Figure 7.7: A 0 to 5 number line.

Just less than 60% of the children made no response to this question whilst a further 8% gave responses that could not be easily classified, for example annotating each number with the letter m, presumably to stand for metres (Child 9, Y5), turning each whole number into a fraction so that the numbers became 1/1, 1/2, 1/3, 1/4, 1/5 (Children 1 and 5, Y5) or simply making marks between each interval without any additional numbers (Child 9, Y4). Such responses were identified as "Other".

Table 7.4 illustrates the percentage of these categories within each year group in the context of the successful addition of numbers to the line.

	Year Group			
Additional Number	Y3 (n=18)	Y4 (n=19)	Y5 (n=16)	Y6 (n=19)
None	94	62	56	26
Other	0	11	19	5
Fractions	6	27	25	21
Decimals	0	0	0	48

Table 7.4: Placing additional numbers on a marked 0-5 number line (%)

It can be seen from the table that the percentage of no response (None) within each class declined with the increasing age and experience of the children as identified from their year group, but even within Year 6 we see that 26% of the class did not give a response, but one child (representing 5% of the class) gave a response that was illustrative of the inclusion of decimals and fractions (Child 2). Because it did not easily fit into the classifications of fraction or decimal because these classifications indicated the single use of one or the other, this response, since it came from only one child in the sample, was also classified as 'Other'.

The remaining 32% of the responses were identified from either the inclusion of 'Fractions' or the inclusion of 'Decimals'. The latter, identified from almost a half of the children (9) within Year 6, were not apparent within the responses of any other year group. Five of these children identified the decimal halves within each interval, but two did not include any calibration. One child (Child 11) simply marked the middle of the full line as 2.5, one (Child 6) began to mark decimal intervals within the interval 0 to 1 whilst Child 17 marked a series of repeated decimals identified within a relative interval, 0.1, 0.5, 0. 9; 1.1, 1.5, 1.9; 2.1, 2.5...

All of the remaining additional numbers involved the use of fractions. One child from Y3 labelled the halves, 1/2, 1 1/2, 2 1/2, 3 1/2, 4 1/2 (Child 1) but 85% of the children who added numbers in fractional form also did this. With the exclusion of a child from Year 1, the distribution was almost numerical equal amongst the other three year groups (4 in each of Years 4 and 5, 3 within Year 6) although Child 1 in Year 4 gave the first additional number as 1/4 but continued by writing the halves. Child 10 (Year 6) started by marking 1/2 but then continued to reform the whole numbers as 1/1, 2/2, 3/3, 4/4, 5/5. Child 3 in Year 4 correctly added 3 1/4 and 4 1/4.

The two questions associated with Figures 7.2 and 7.7 were related to number lines that had an identical structure in terms of intervals. The first question gave children the freedom to create the number line of their choice, whilst the second confined them to thinking beyond the marks and numbers visible on a particular number line. Though the majority of children were able to do the former, it was a minority, just over 40% who responded to the latter. It is possible that a clue to this low response rate is seen from the dominant role that marking whole numbers played when completing the calibrated line. Only within Year 6 do we find more than one quarter of the children within any one year attempting to add additional numbers. This would add further evidence to suggest that in general, the embodiment of the number line held by most children below Year 6, is one in which any markings are represented by whole numbers. There is evidence to suggest that this is slowly changing from Year 4 with some children including fractions and within Year 6, we also see the use of decimals. Evidence that whole numbers, fractions and decimals may be included on one line was very limited.

7.3 Children's Conception of the Number Line

The evidence presented within §7.2 suggests that children's embodiments of the number line are associated with a horizontal or slightly diagonal line with numbers ascending left to right. The evidence also suggests that for the greater proportion of children, lines should be marked with whole numbers, whilst a minority, generally older children, recognize that intervals may be partitioned. It was more usual for these intervals to be partitioned into two with a consequent identification of the half, either as 1/2, or as seen amongst older children as 0.5. Within this section, we explore from the results of interviews with selected children, their conceptions of the number line.

The children selected for interview (see §4.4.3) were each asked, "What is a number line?" The responses of the children were most frequently descriptive and either associated with descriptions of what may be identified as personal embodiments, particular number lines within their classes, or of actions associated with the number line.

Over 75% of the children, largely from within Years 2, 3 and 4 but with single children from Years 5 and 6 gave responses that matched the evidence apparent within (§3.3.2) in that they were either descriptions of some features of a number line or were associated with some sort of action. As was noted within the Pilot study (§3.3.2) only one child provided a response that differed qualitatively from the more common responses in that, whilst it contained description and was associated with actions it also included reference to decimals as well as whole numbers:

> A number line is just a line that put one number like sixty-seven (Draws line and puts 67 at left end) and one number say like seventy (marks right end), and the numbers that go in between. You put them, just like, so then you'd have sixty-eight there, sixty-nine there and... sixty-seven, sixty-eight, sixty-nine and in between you have decimals like sixty-seven point five, then you'd have... if you wanted to go up in naught point fives and you'd have sixty-eight and then sixty-eight point five and so on. It's just to help you work out the take-aways or addings basically... or if you got one hundred and seventy-three (draws another line and marks left end) and you want to get to one hundred and ninety (right end), it would help you, coz then I could add seven (makes jump) and that would be one hundred

> and eighty and then I could just add ten (makes jump) and that would get me to that. It helps me in my take-aways and adding. (Child 6.2)

The descriptions that the children provided for their definitions of the number line either related to specific lines suggesting a particular embodiment of the number line, or they were associated with memory of activities the children had seen or carried out with the line. However, whichever emphasis the children placed on their 'definition' they also included reference to the number line as a tool.

> They are things you have to put lines... and you've got numbers underneath the lines (Draws a line starting at 0 and marks the units to 19) You can count on it. One, two, three, four, five... up to one hundred. (Child 2.1)

> It's a big row of numbers, just straight... it's going that way (towards the right) instead of that (top to bottom) way. Sometimes you can get number squares. It's the same as a number line, but it's a line. I'll go and get one... They tell you how to count. (I: What are they made of?) Wood. Sometimes wood. (Child 2.2)

> We had this long number line and we had a blank thing... we stuck some numbers on each end. (I: What was the blank thing?) A ruler. (Child 3.1)

> Numbers... Counting back and forth. (Child 3.3)

> A number line is a line that has numbers on. (Child 4.2)
>
> It's when you can put numbers on line. So you start with the lowest number first... and then you start with the biggest number after. At the end. (Child 4.3)

> A number line is where it helps you work out add and all that. So you can put zero there... and then put one hundred there... and then we want twenty, thirty... (Child 5.1)

> Just a line that you can count on. (Child 5.2)

> Just a line that you count numbers on ... you put numbers on and you had to take things away and add things and times and that. (Child 5.5)

> It's like a line that you can add up on, to make it easier for yourself. It is like... then you can take like jumps... or you can just say it was fifty and your line goes to sixty, then you can do five and that would get you to your fifty-six and then you can do four (hand movement)... Which number should I do... how should I add it up... and what would my answer turn out to be, coz you can do take-aways on number lines as well. (Child 6.1)

> You got a line. It's like a big line and then you got a number at the start and then a number at the end and then you got all the little numbers in the middle of it. If you want to make a number you can do the jumps on the number line (hand movement). (Child 6.3)

> It's like a line and you put your starting number like if you put eight and you want to get to sixteen, there's like... you do loads of jumps until you get to the end and you put sixteen at the end (hand movement)... I need to get to eighteen ... (Child 6.4)

> A number line is what helps you count on in numbers. Say if you had to get from fifty-six to one hundred... it would help you. Say get fifty-six and get it to the nearest ten. Add four on to get it to sixty, then you can just add the forty. (Child 6.5)

There was no surprise in that the quality of the responses to this question reflected the way that the children had been chosen (see §4.4.3; Table 4.2). Child's 6.2 responses to the four item-words (see §4.4.3) were 75% Specific but despite this he was classified as a high achiever by both his teacher and the SATs results, and he confirmed this by providing the highest quality of responses throughout the study. The majority of the children who provided the above 'descriptive' quotes (of episodic and specific nature) expressed either 'specific', 'episodic' or 'specific/episodic' overall response to the four item-words asked. Only children 5.2, 6.1 and 6.4 responses to those words were identified as 'generic', indicating a more relational way of thinking, which encloses 'specific' and 'episodic' responses (like those expressed above) as well as responses of higher quality. What was a surprise was that in essence there was little qualitative difference between the 'definitions' provided by children of different year groups. References to counting for example were provided by a Year 2 Child (2.1) and a Year 6 child (6.5). What was also noticeable were that descriptions in many instances replicated statements used by teachers to clarify actions — example Child 4.3 and the opening comments of her Y4 teacher (§6.2.4).

7.4 Pinpointing Numbers on the Number Line

The NNS suggests that children should be able to associate inherent knowledge of the number system from 0 to 100 with magnitude to estimate the position of numbers on a 0 to 100 number line. Throughout the early stages of the NNS there are recurring references that children within Years 2 to 4 (median ages 6.5 to 8.5) should be able to have experiences that would enable them to interpolate the position of a particular number on a partially numbered or an empty number line, record their estimates and find the difference between the estimate and the actual number. For Year 2 children the presented examples include number line segments with the ends marked 0 and 10 and for Years 3 and 4 number line segments marked from 0 to 100. It is implicit that children within the Years 5 and 6 possess the skill and understanding to do this.

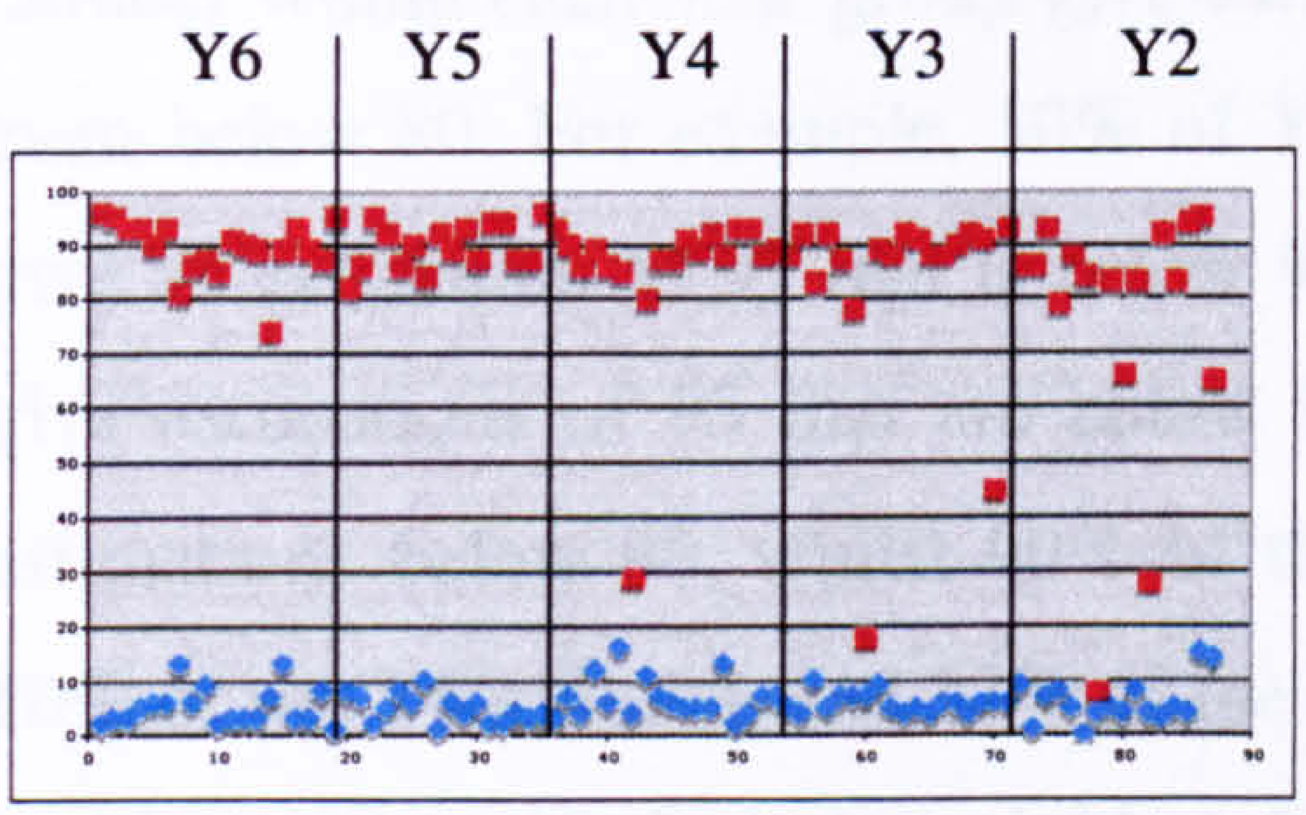
This section reports on the quantitative outcomes derived from the presentation of a series on 0 to 100 number lines with the invitation that the children place particular numbers on the number line (§4.3.3.1). The individual items were compatible with items identified from within the NNS (See Section 5; pp. 8, 9, 17) but were also selected to consider differences in the children's ability to estimate numbers that were equidistant from the end points 0 and 100. The numbers, in the order that the children responded to them were 93, 45, 12, 5, 75, 3, 7, 25, 97, 95, 88 and 55. This was inspired by a comment within QCA (1999a):

> An important aspect of having an appreciation of number is to know when a number is close to 10 or a multiple of 10: to recognise, for example, that 47 is 3 away from 50, or that 96 is 4 away from 100. (p. 28)

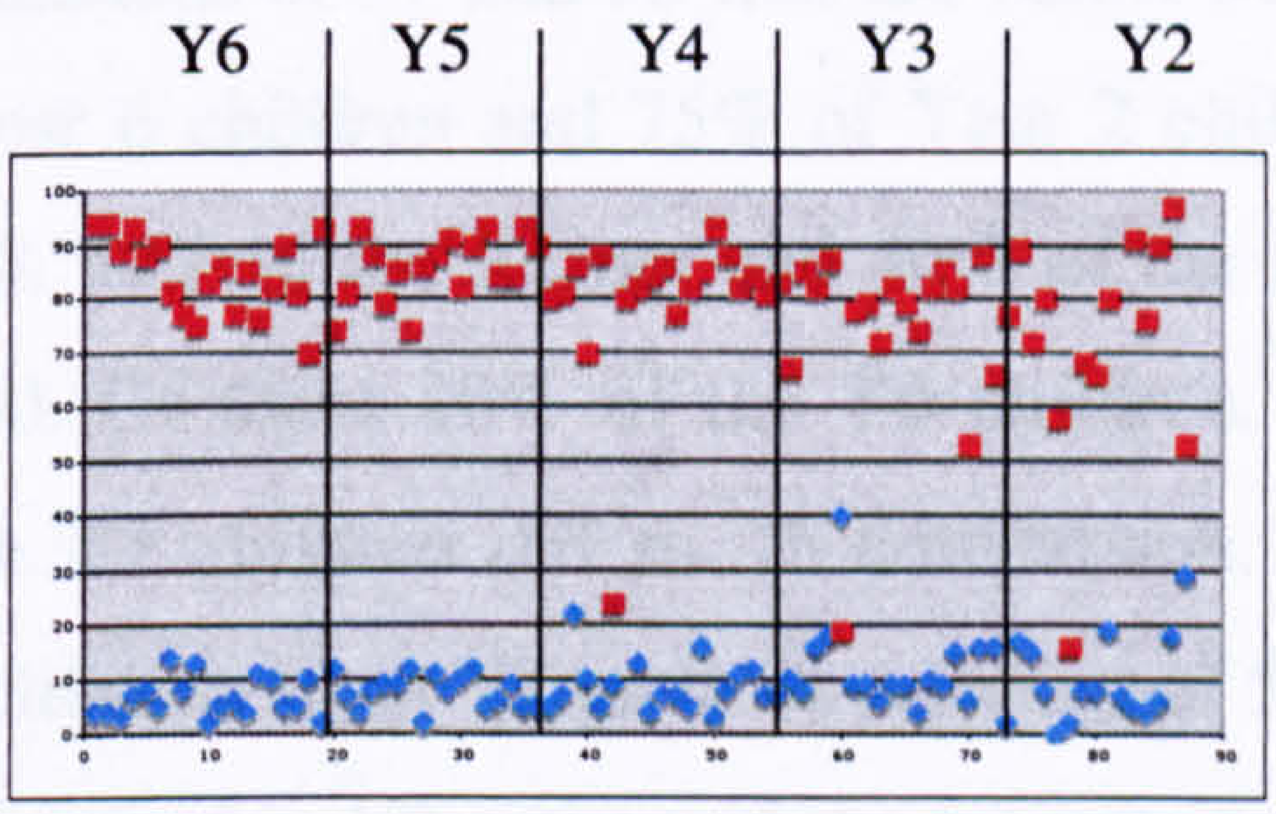
The sample responding to the pinpointing questions consisted of 90 children (18 from Year 2, 18 from Year 3, 19 from Year 4, 16 from Year 5 and 19 from Year 6).

7.4.1 Estimations of Magnitude

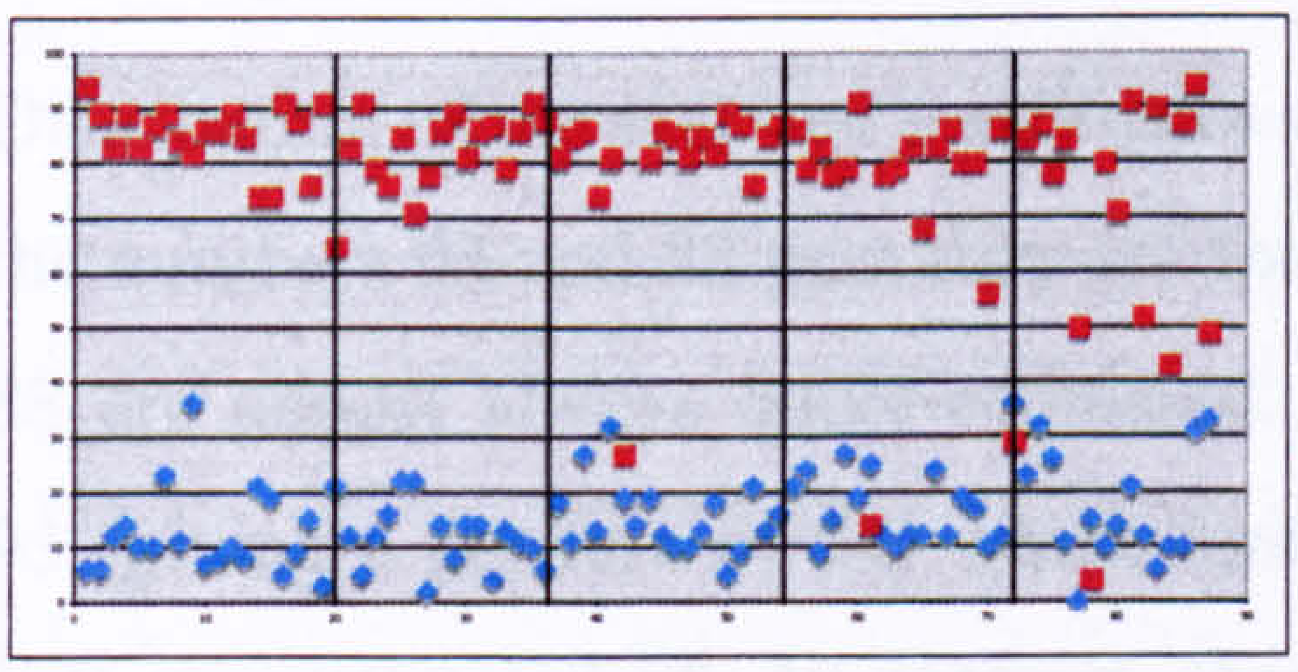
Figure 7.8 provides an illustrative example of the way of the distributions of the estimations associated with each pair of numbers. The figure is constructed by providing all of the estimates from all of the children in the sample and the distributions associated with each year group may be identified within the relative columns of each graph. In each instance the horizontal scale is marked to 90 (the total number of children) whilst the vertical, to maintain consistency, is on a 0 to 100 scale subdivided into units of ten.



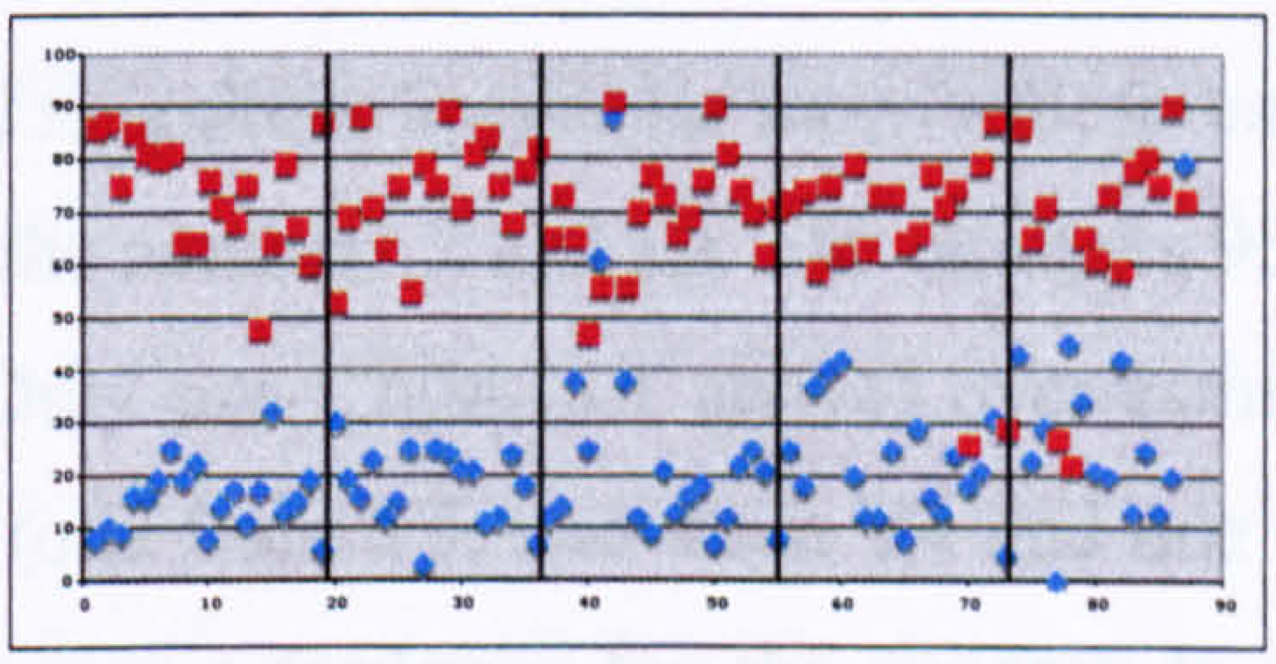
(i) Distribution of 3 and 97



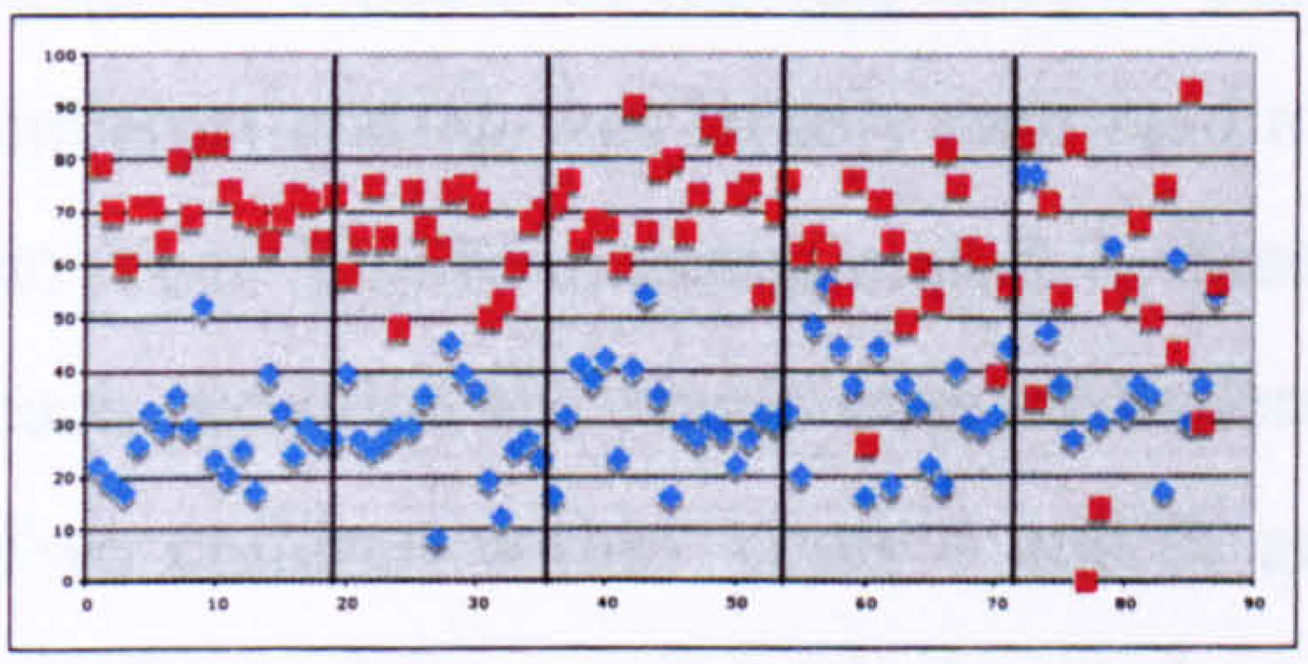
(ii) Distribution of 5 and 95



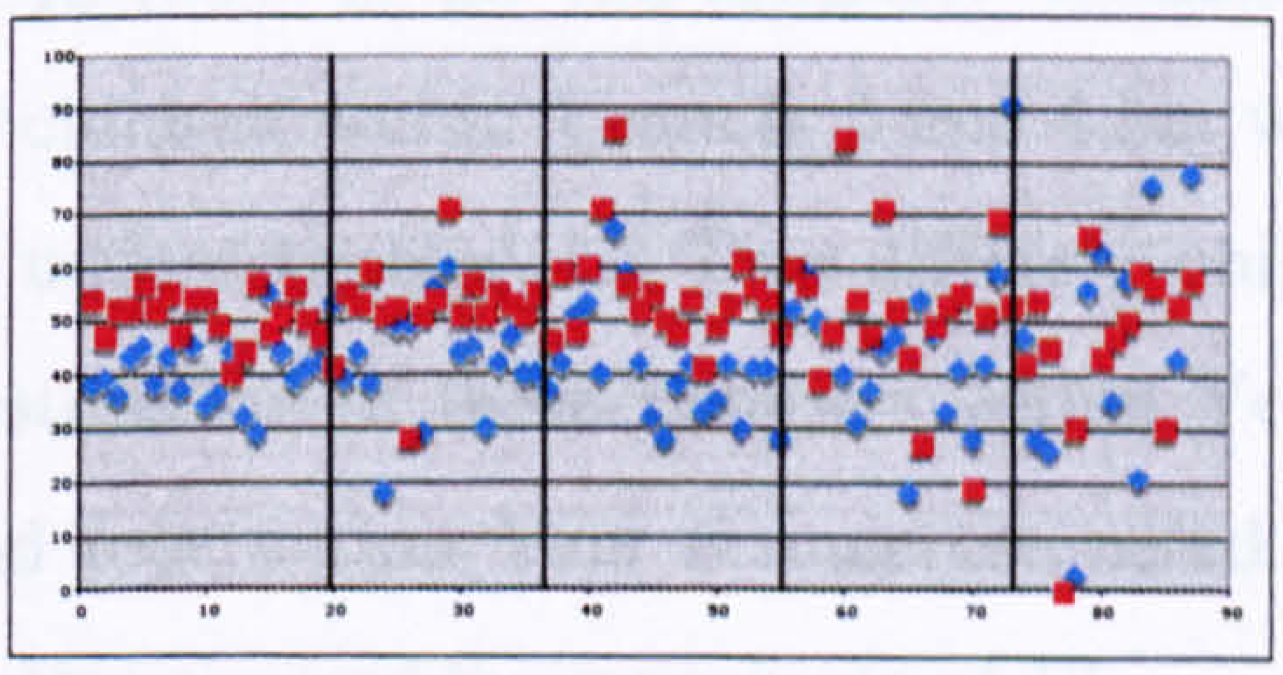
(iii) Distribution of 7 and 93



(iv)Distribution of 12 and 88



(v) Distribution of 25 and 75



(vi)Distribution of 45 and 55

Figure 7.8: Comparative distribution of estimations associated with number pairs, by children within Y2 to Y6

From the figure, it can be seen that:

- The overall accuracy of the estimations by the whole group of children decreases as the magnitude of the number to be estimated moves away from zero.
- Although accuracy tends to improve as the numbers move closer to 100, for example 97 and 95, a noticeable number of individual children, particularly from within Years 2, 3 and 4 display a remarkable degree of inaccuracy, whilst a considerable number within each year group give estimations of 97 and 95 that are below 90 and even below 80. For example, 50% of Year 6 children and 75% of Year 2 children give an estimation of 97 that is below 90. 70% of the former and 80% of the latter give estimations of 95 that are below 90. Of these 40% of the Y6 children, give estimations below 80, whilst 90% of the Y2 children did so. In comparison, only 10% of the Y6 children and 27% of the children from Y2 gave an estimate of 5 that was larger than 10. Only two children from within Y2 estimated the magnitude of 5 to be larger than 20.
- Distribution of the children's estimates of magnitude appear far more widespread for the numbers 93 and 88 than they are for the numbers 7 and 12. The numbers 93 and 88 are almost always under-estimated. Only one Child (6.1 above) over-estimated 93 (giving a position of 94) whilst four (one from Y5, two from Y4 and one from Y2) overestimated 88. In contrast, there was a tendency for the children to over-estimate 7 (88% of the full sample) and 12 (80% of the full sample). Evidence of underestimating was largely restricted to children within Years 6, 5 and 4 but whilst no Year 3 child underestimated 7, three underestimated 12. Two different children each provided the single case of underestimation of these numbers within Year 2. Two children within Years 4 and 5, and four within Year 6 underestimated both numbers whilst two in each class only underestimated the magnitude of 12.
- The distributions associated with 25 and 75 appear almost random, although each estimate is generally confined to an appropriate half of the 0 to 100 line. Again, the main distinction between the two numbers is generally a tendency to overestimate 25 and underestimate 75. 14% of the full sample, with almost half of this proportion

from Year 4 but none from Year 5, over-estimated 75, whereas 28% underestimated 25.

- The distributions associated with estimations of 45 and 55 are interesting, given the context of the above discussion. Though there is the possibility to provide either an underestimate or an overestimate for these numbers, the distribution of the estimates would seem to have more in common with the distributions for 97 and 3 than for distributions associated with other number estimates. Pinpointing these numbers does not have the limitations that are associated with pinpointing 7, 5, 93 or 95, there are no upper and lower limits and yet there is a sense that the children were being guided to hone in on the numbers. Over 60% of the children pinpointed 55 to within ±5 units of accuracy. 40% did so to 45. Children within Y5 demonstrated the greatest accuracy, 82% identifying 55 to within ±5 units and 54% doing so with 45. Interestingly, all of the other year groups were also relatively accurate with 55 with ±5 degree of arranging from being achieved by 63% of Year 6 to 43% of Year 2 although this range decreased from 40% (Y6) to 15% (Y2) when 45 is considered.
- Absolute accuracy, defined as an exact estimation, was only achieved in 4% of the 1080 estimates that were considered. Almost 50% of these were identified in the instances where 3 and 5 were considered and absolute accuracy was identified equally for both of these numbers. Only two others, 12 and 55, each contributed 15% to this 4%. No absolute accuracy was identified from estimates of the numbers 93, 95 or 97.
- A more generous interpretation of accuracy has been used to illustrate distinctions in the ways that the children estimated the number magnitudes. This notion of accuracy is identified using a range of ±2 units from the actual number. Almost 20% of the responses of the full sample achieved this degree of accuracy in some of their estimates but it is noticeable that three numbers account for 50% of this proportion: 3(22%), 5(16%) and 55(13%). The remaining 50% is accounted for by the other nine numbers of which the lowest proportions are 93(2.5%), 97(3%) and 88 (4%). Not unexpectedly, the greatest accuracy was noted in the responses of children from within Y6 where 28% of their responses satisfied this degree of accuracy. The

lowest proportion was noted from the responses of children within Y3 (10%) and Y2 (13%).

The characteristics identified from Figure 7.8 are shown in sharper relief, if each year group is highlighted and if we draw on distributions that are associated with differences between estimations and actual numbers. It is to the first of these two considerations that we now turn.

7.4.2 Year Considerations

Figure 7.9 provides an illustrative example of the accuracy of estimates that the children from each of the five year groups gave to the numbers 88 and 12.

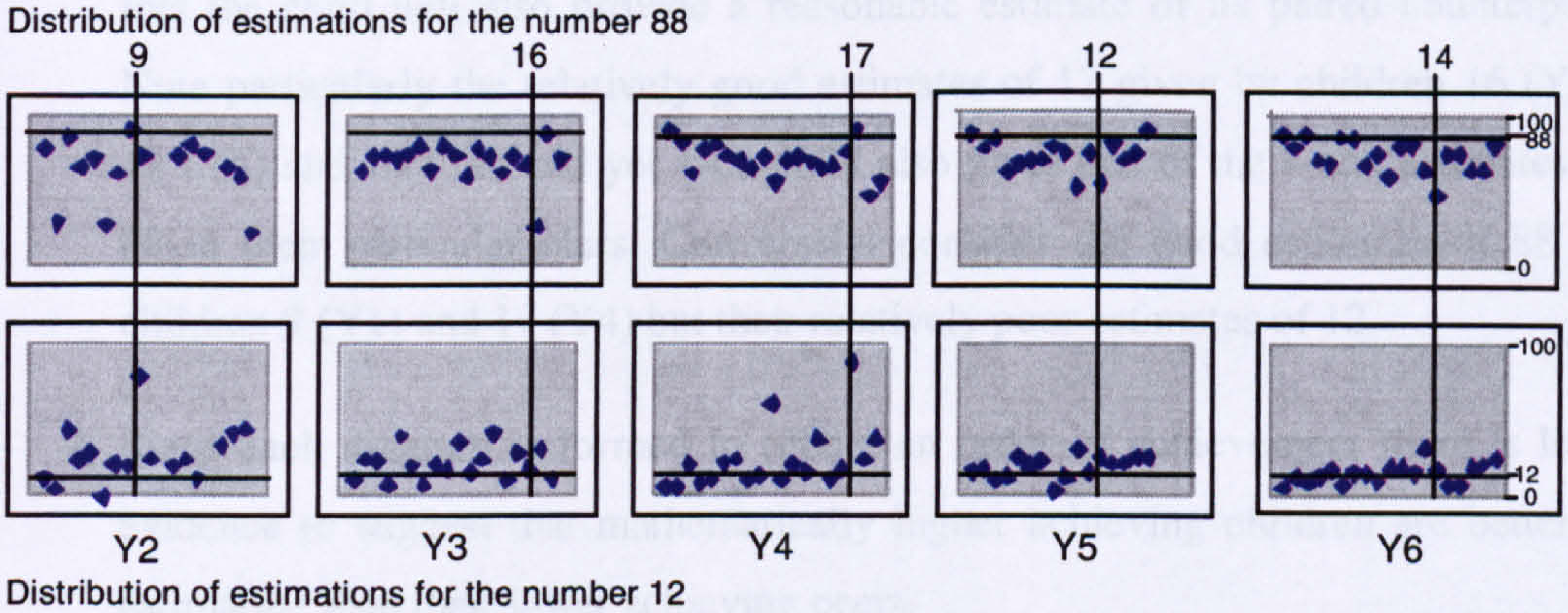


Figure 7.9: Full sample estimations of 12 and 88 identified by year group

Each child's estimate of the numbers 88 and 12 (marked on the figure) is identified on a vertical scale marked 0 to 100 and each point illustrating the distributions is arranged so that children within each year group are given in an order that reflects their overall achievement within mathematics as measured by inter school tests and teacher predictions of achievement within Standard Attainment Tasks (see also §4.4.3 and §7.3). High achievers are to the left and low achievers are to the right.

The numbers 9, 16, 17, 12 and 14 represent particular children whose estimates for the two numbers may be compared through the vertical lines.

Even though the numbers are equidistant from the end points of the 0 to 100 line, several features emerge from Figure 7.9:

1. A general trend amongst the majority of children is to underestimate the position of 88 and over-estimate the position of 12.

2. Even though there may be a wide spectrum of estimations within a particular year group, their accuracy tends to be associated with age. Estimations of the older children, particularly those for 12, are generally more accurate than those of the younger children. Though a gradual convergence is noted for the estimate of 88, even amongst the older children there is a tendency to extensively underestimate.

3. A child's ability to provide a reasonable estimate of one number does not imply that the child will also provide a reasonable estimate of its paired counterpart. Note particularly the relatively good estimates of 12 given by children 16 (Y3), 12 (Y5) and 14 (Y6) and yet each child also gives one of the worst estimates of 88 in their particular class. Conversely, consider the good estimates of 88 by children 9 (Y1) and 17 (Y4) but their relatively poor estimates of 12.

4. Since each diagram is formed to reflect an order of achievement there is little evidence to suggest that mathematically higher achieving children are better at estimating than their lower achieving peers.

A more comprehensive examination of these points can be made through a comparison of each year group's overall estimates of all numbers. The results from each year group to the range of numbers is presented within Figures 7.10 and 7.11, the first illustrating estimations of numbers below 50 and the second illustrating estimations of numbers above 50. Both figures are formed in the same way as Figure 7.9 but for simplicity exclude some of the information on that figure although estimations for individual children can be traced by considering the vertical orientations. The impressions left by Figure 7.8 and 7.9 are more clearly seen through Figure 7.10

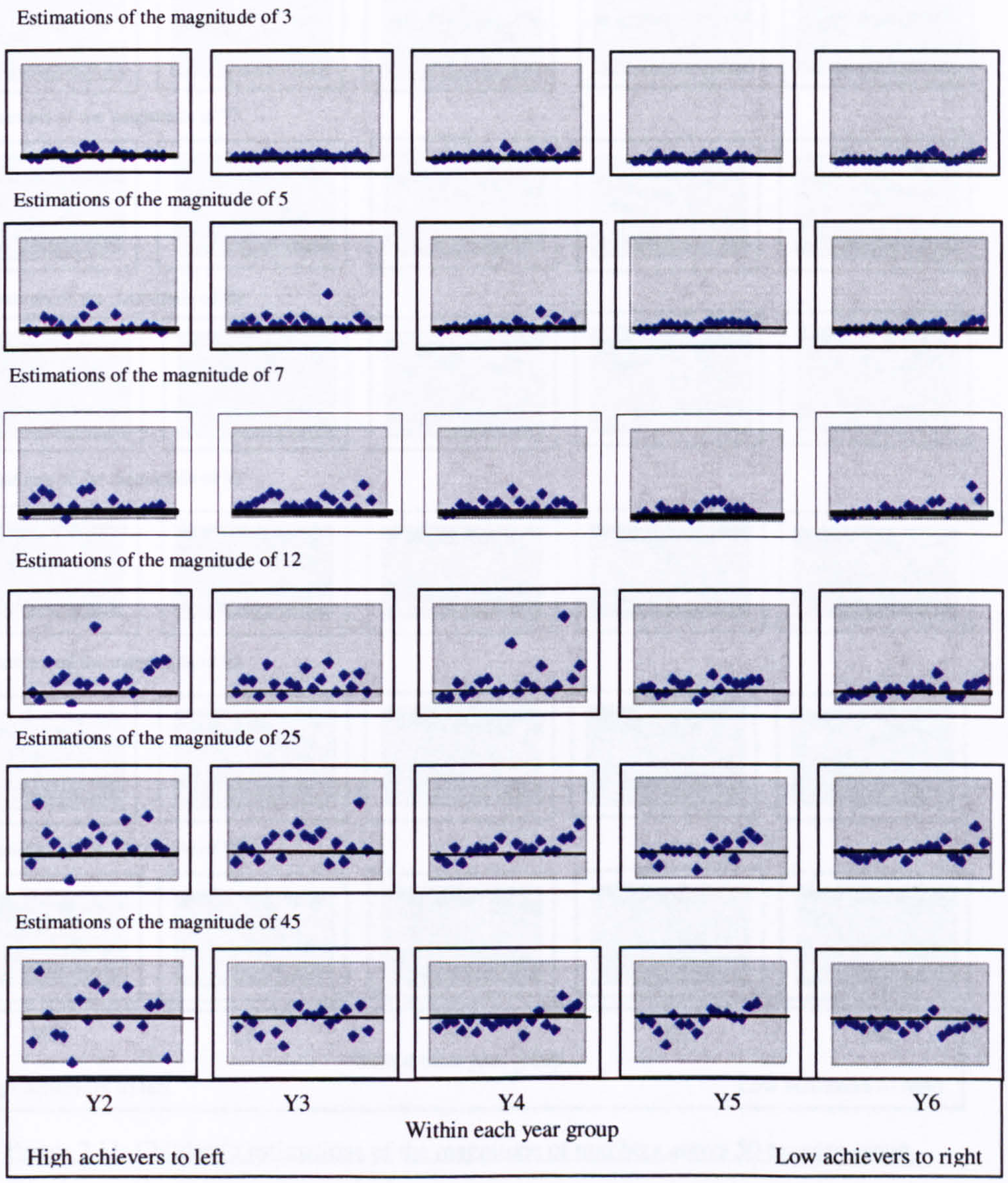


Figure 7.10: Children's estimations of the magnitude of numbers below 50 by year group.

Figure 7.10 illustrates:

- the overall view that accuracy in estimation increases with age.
- the accuracy of estimation is higher with low numbers than with high numbers.
- there is little consistency in individual children's accuracy of estimation.
- there is a general tendency to increasingly over-estimate the value of numbers less than 45 as they become more removed from zero.

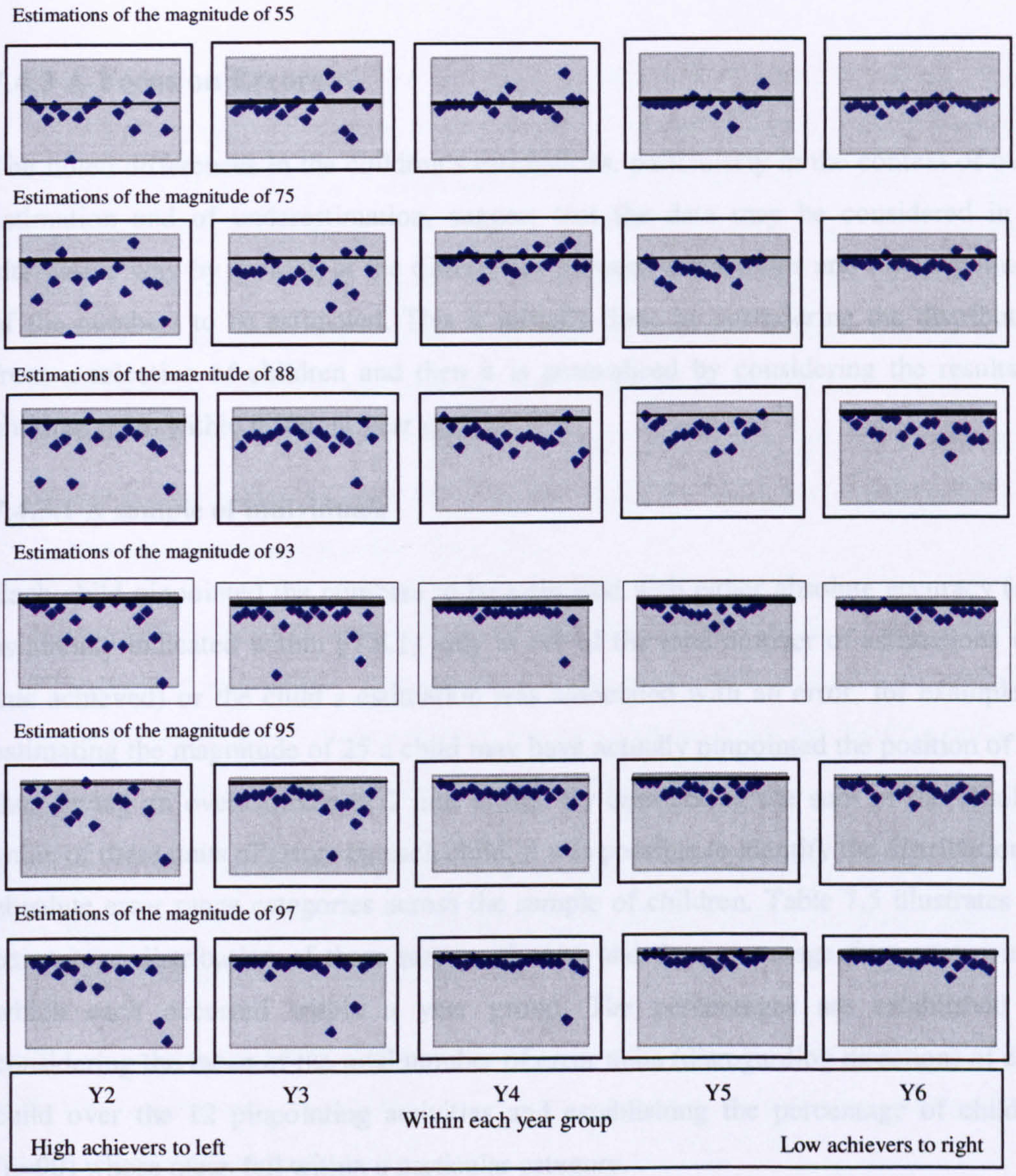


Figure 7.11: Children's estimations of the magnitude of numbers above 50 by year group.

Figure 7.11 indicates the children's estimates of magnitudes above 50. All of the features apparent within Figure 7.10 are again clearly apparent within Figure 7.11 with two exceptions:

- All of the numbers are generally underestimated.
- The nearer numbers are to 100, then generally there is an improvement in accuracy, although 95 appears to be something of an anomaly, whilst most children within Years 4, 5 and 6 appear to be relatively accurate in pinpointing the position of 55.

7.4.3 A Focus on Errors

The noted differences in the children's estimations, particularly in the context of over-estimation and of underestimation, suggest that the data may be considered in an alternative way by looking at the differences between estimations and the magnitudes of the numbers to be estimated. This is initially done by considering the distribution from a selection of children and then it is generalised by considering the results of children from within different year groups.

7.4.3.1 A sample of individuals

Each child pinpointed the numbers to be estimated with either absolute accuracy (and as already indicated within §7.4.1) only in 4% of the total number of estimations was this achieved) or the child's estimation was associated with an error, for example in estimating the magnitude of 25 a child may have actually pinpointed the position of 27, thus giving an overestimate of 2 unit errors. By considering the sum of the absolute value of these units of errors by each child, it was possible to identify the distribution of absolute error range categories across the sample of children. Table 7.5 illustrates the percentage distribution of these ranges of error and the percentage frequency within which each occurred within a year group. The percentages are established by considering the mean of the total number of error units (disregarding direction) of each child over the 12 pinpointing activities and establishing the percentage of children (n=90) whose mean fell within a particular category.

	Units of Error (e)				
Year	$0<e\leq5$	$5.1<e\leq10$	$10.1<e\leq20$	$20.1<e\leq30$	$30<e$
2	0.0	1.1	12.6	2.3	2.3
3	1.1	10.3	6.9	3.4	1.1
4	3.4	11.5	3.4	1.1	0.0
5	2.3	12.6	3.4	0.0	0.0
6	9.2	8.0	3.4	0.0	0.0
Total	16.1	43.7	29.9	6.9	3.4

Table 7.5: Percentage distributions of the mean error range within different year groups

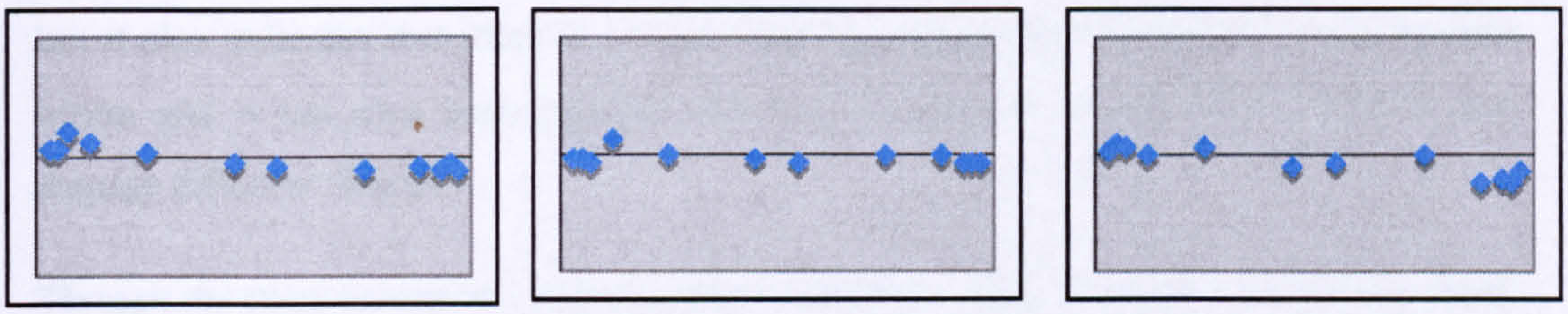
The indications from Table 7.5 are that for the range of numbers that formed the pinpointing component:

- The accuracy of almost 60% of the children, only one of whom was from Year 2, fell within a range of 10 error units. Of these 27% (14 children), mostly from Year 6, achieved an accuracy of 5 error units. Earlier it was noted that almost 20% of the responses of the full sample achieved a ± 2 degree of accuracy with particular numbers. Children whose overall levels of estimation provided a mean absolute error of 5 may have approached this level of accuracy (given that the former considered direction but the later disregards direction). Indeed the absolute mean error of one child from Y5 (Child 5.3) was 1 unit. Only her estimation of 12 exceeded 2 and that was an over-estimate of 4. She achieved absolute accuracy pinpointing three numbers (88, 75 and 25).

- Estimations from children within Y2 provided the greatest degree of absolute error. 90% of their estimations were above 10 error units, 10% of these (two children) being above 30. The overall evidence suggests that the older the children were, the lower the mean absolute error associated with their estimations.

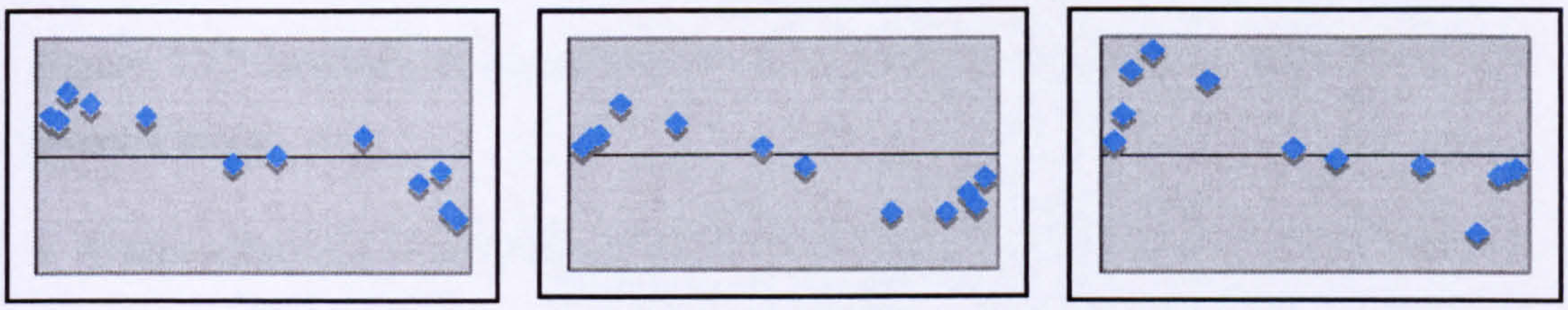
Looking at the sum of the absolute errors, each individual, provides the basis for selecting particular individuals and considering the distribution of the errors associated with each of the numbers pinpointed (Figure 7.12).

Figure 7.12 is established by considering the direction of errors associated with individual estimations. The children were selected to indicate typical responses within the first three categories of absolute error.



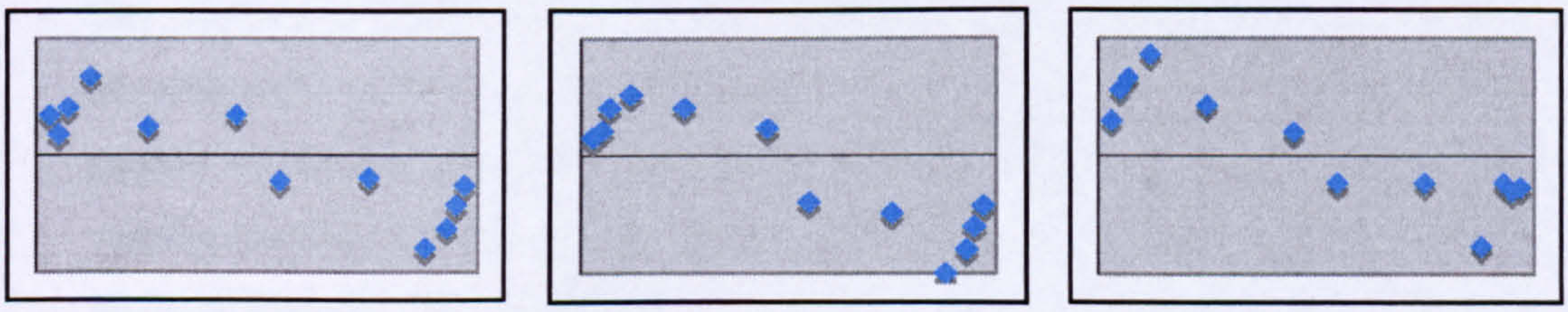
CHILD 6.4 CHILD 5.3 CHILD 4.16

(i) Distributions associated with 0 – 5 absolute error units



CHILD 6.7 CHILD 3.10 CHILD 2.4

(ii) Distributions associated with 5 – 10 absolute error units



CHILD 6.15 CHILD 5.1 CHILD 4.4

(iii) Distributions associated with 10 – 20 absolute error units

Figure 7.12: Illustrative individual estimates of magnitudes.

It may have been possible to consider these distributions by associating the actual scores of each child with a line of best fit, thus attempting to identify a regression model. But given the nature of the distribution and the general interpretations that suggest a range of numbers that are under-estimated and a range of numbers that are over-estimated it was decided to move away from attempting to identify the characteristics of the children's pinpointing estimates through such a model. Within the restricted number line, children are unable to provide an estimation below 0 or above 100. These two points act as limits upon which estimations, particularly of numbers at extremes, would seem to converge. The evidence previously considered indicates this, but it also indicates that there is a wide range in estimations of numbers between these limits and it has also been implied that the estimations of individual children may display different features.

Though the three groups of children reflect different categories of estimation, the trends in their responses to estimation appear to have similar characteristics. It is only in the degree of error associated with the groups that the characteristics may change.

Figure 7.12 confirms at the individual level what has been earlier suggested at the general level:

- Underestimation of numbers within the first half of the number line — and here the notion of half should be taken a little liberally since several of the children's estimates of 45 may not satisfy this conclusion.

- The children identified within Figures 7.12 (ii) and 7.12 (iii) appear to treat 45 and 55 differently to the way the numbers that fall before and after these numbers are treated. On the one hand, the numbers between zero and 45 are over-estimated, with varying degrees of error, whilst on the other the numbers between 55 and 100 are under-estimated. The numbers 45 and 55 are generally associated with a convergence towards a smaller error than the more extreme errors associated with numbers within the inter-quartile range of each half of the line.

Distributions associated with the mean of the sum of the absolute errors (Mean Error) associated with each number from within each year group are considered within Table 7.6.

		Year Group									
		Y2		Y3		Y4		Y5		Y6	
	Actual Number	Mean error	Direction	Mean	Direction	Mean	Direction	Mean	Direction	Mean	Direction
	3	3	(+)	3	(+)	4	(+)	3	(+)	3	(+)
	5	6	(+)	7	(+)	4	(+)	4	(+)	3	(+)
	7	11	(+)	11	(+)	8	(+)	7	(+)	6	(+)
	12	18	(+)	11	(+)	14	(+)	8	(+)	6	(+)
	25	18	(+)	14	(+)	9	(+)	8	(+)	6	(+)
	45	21	(+)	11	(-)	9	(-)	8	(-)	6	(-)
	55	9	(-)	11	(-)	6	(+)	6	(-)	5	(-)
	75	22	(-)	16	(-)	7	(-)	10	(-)	6	(-)
	88	25	(-)	19	(-)	18	(-)	15	(-)	14	(-)
	93	23	(-)	20	(-)	17	(-)	11	(-)	8	(-)
	95	23	(-)	20	(-)	15	(-)	10	(-)	11	(-)
	97	21	(-)	15	(-)	12	(-)	7	(-)	8	(-)
sum		200		158		123		97		82	

Table 7.6: Means of absolute errors and the sum of mean errors as a percentage of the sum of the magnitudes to be identified.

Table 7.6 also illustrates the direction of the errors, whether an under-estimate (-) or an over estimate (+) each calculated by finding the mean of the actual estimations of children within a particular year. This consistency of direction, apart from that of 45 and 55, was generally clearly obvious from the raw data (See Appendix VI). As an illustrative example, the absolute mean of the estimates of the children within Y3 for 45 was 41, so this was identified as an underestimate. Similarly, the mean estimates of 45 for the other year groups apart from Y2 were also regarded as underestimates, the lowest mean estimate being 40 from the children of Y6. Only in one instance, Y5, were the estimations for 55 regarded as an over-estimate.

Clearly, the means of the sum of the absolute errors decrease as the children from the older classes are identified. Within Y6, this is two and a half times smaller than that identified from the children within Y2. Interestingly, though the difference between the mean of the sum of absolute errors for the magnitude of 3 is generally equal between each year group we note that within all year groups there is a general tendency for the means to increase as the magnitudes to be estimated become larger. There are of course some exceptions to this rule:

- There is a remarkable consistency in the mean errors of the children within Y6 until they need to estimate numbers larger than 75.

- Within all year groups, apart from Y3, the mean error associated with the magnitude of 55 is less than that associated with the preceding and the subsequent numbers.

Diagrammatic illustration of the distributions of the mean of the sum of absolute errors for Y2 and Y5 illustrates the general trends that may be observed within Table 7.6, which will serve as a focus for discussion within Chapter 8. Figure 7.13 illustrates the size and direction of the means of the sum of absolute errors of the estimates for each of the numbers estimated by Years 2 and 5. The figure is constructed by establishing for Y2 (continuous line) and Y5 (dashed line) the mean of the absolute differences between each child's estimate of the position of the number and the actual number and this is then associated with the notions of under-estimation or over-estimation as identified for Table 7.6. The vertical scale illustrates difference between the estimations and the actual numbers.

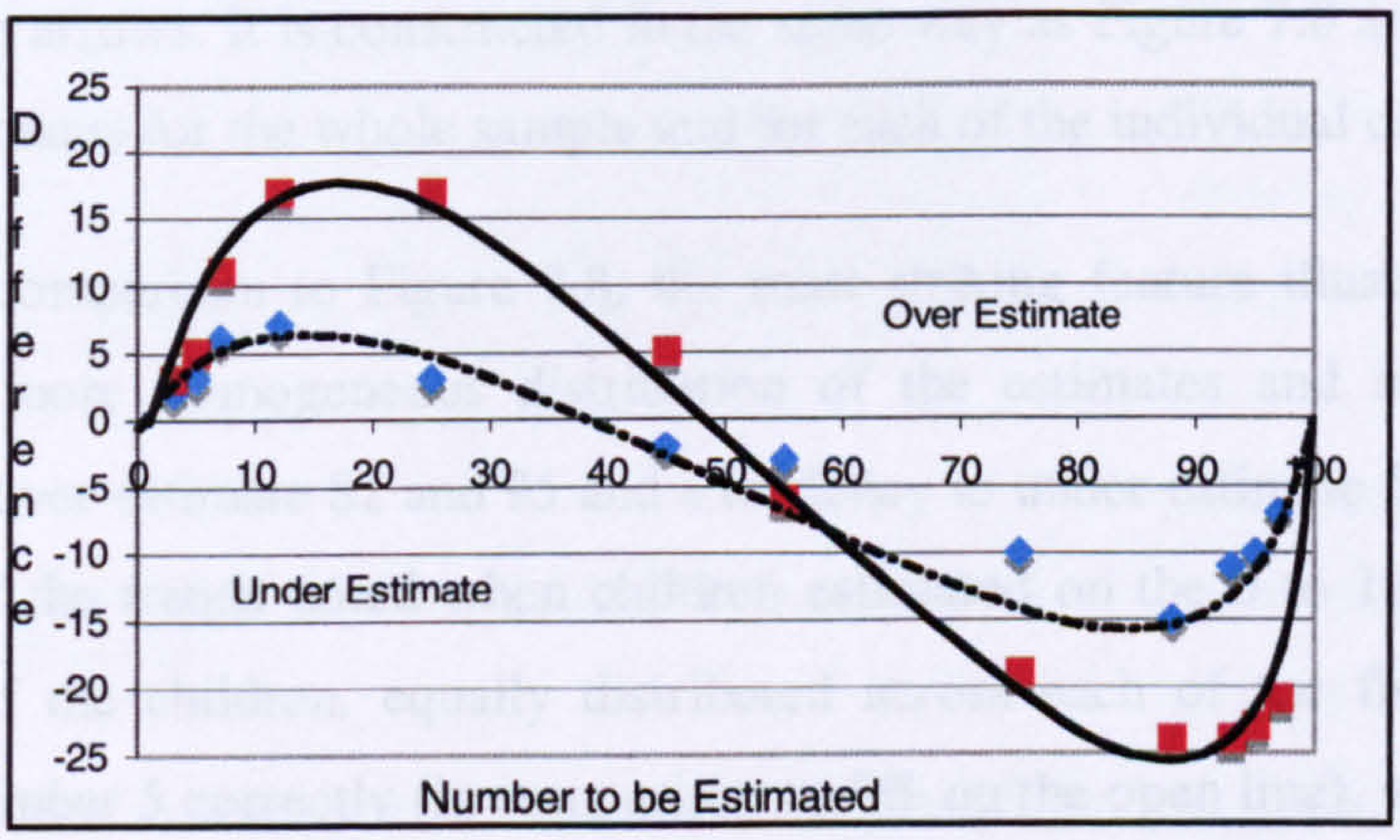


Figure 7.13: Difference between actual numbers on the number line and the mean of the absolute differences between that number and the estimates given by Years 2 and 5.

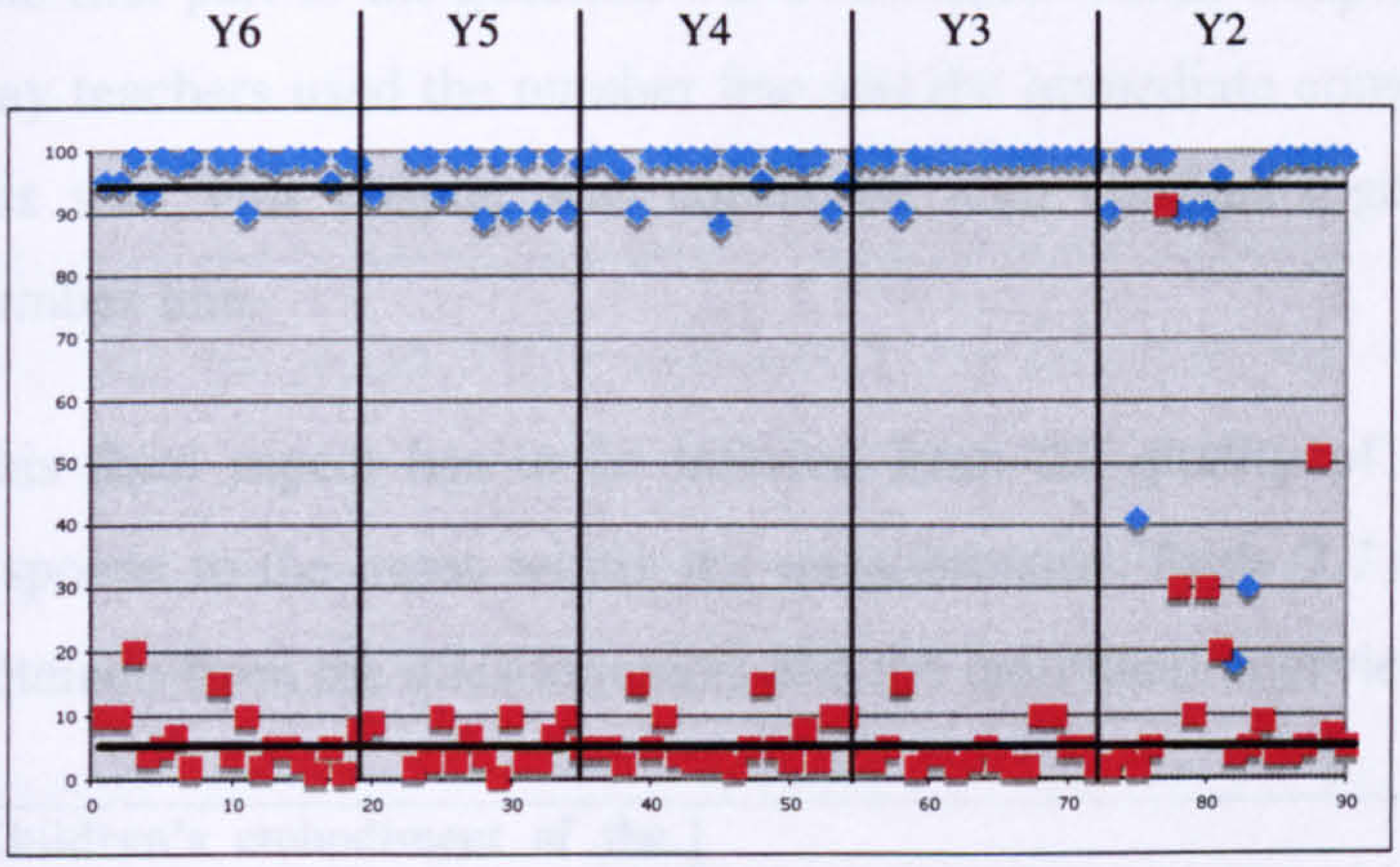
The general features within the figure are characterised by the curve. This is not intended to have significance beyond the descriptive. It is a mechanism that illustrates a summary of the features that have emerged from previous discussions of the distributions of the estimations. Though the figure illustrates the greater accuracy associated with the estimates of the Y5 children compared to those of the Y2 children, it also illustrates the similarity in the nature of the distribution within the two classes — magnitudes below 50 are generally exaggerated, those above 50 are generally underestimated, whilst there would appear to be little relationship in children's ability to estimate magnitudes near the left hand of the number line and their ability to estimate those near the right hand extreme.

7.5 Estimating the Magnitude of Pinpointed Values

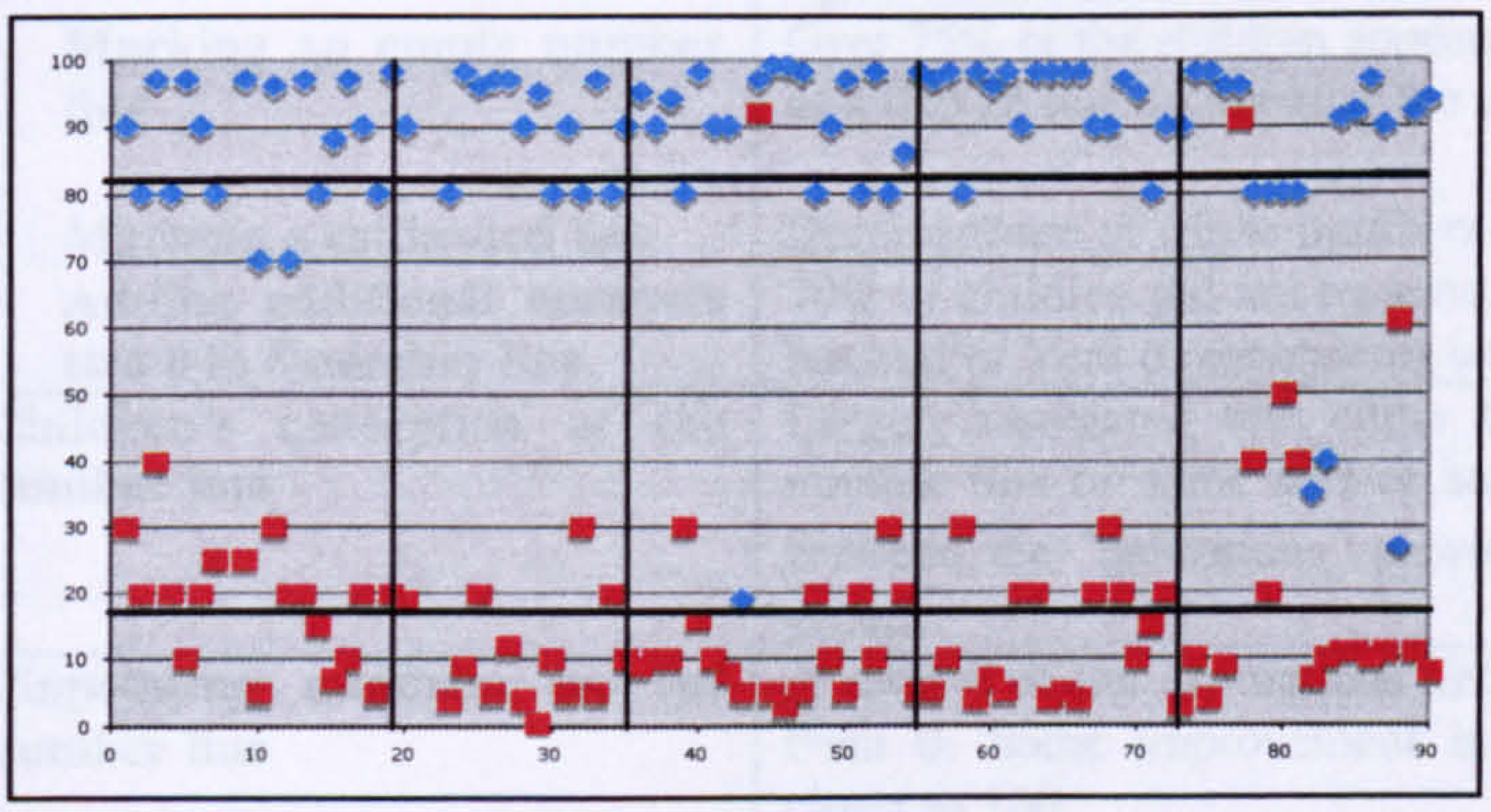
Within this section, the children's responses to the tasks that involve the identification of the number represented by the arrow on 0 to 100 number lines are considered. The numbers represented by the arrows were 82, 18, 95 and 5 (Appendix IV, Qs.5-8), again selected because they are the same numerical distance away from the ends (see QCA, 1999a, p.28). Examples of recognising position of already pinpointed numbers on a number line are also stated within the NNS (Section 5, p. 17; Section 6, p. 11).

Figure 7.13 illustrates the distribution of the children's estimates of the magnitudes shown by the arrows. It is constructed in the same way as Figure 7.8 and illustrates the range of estimates for the whole sample and for each of the individual classes.

Perhaps, in comparison to Figure 7.8, the most striking feature illustrated by Figure 7.14 is the more homogeneous distribution of the estimates and in particular the tendency to over-estimate 82 and 95 and a tendency to under-estimate 5 and 18. This is a reversal of the trends noted when children estimated on the 0 to 100 open number line. 20% of the children, equally distributed across each of the five year groups, estimated number 5 correctly (in comparison to 9% on the open line), whilst 5% did so for number 95 (whereas none did so on the open line). 15% of the children gave a response of 3 to the number 5 whilst 60% estimated that 95 was 99. Children who provided the most extreme estimates were largely situated within Year 2, with four children giving estimates larger than 20 for the number identified by the arrow at 5.



(i) Distribution for 5 and 95



(ii) Distribution for 18 and 82

Figure 7.14: Distributions of estimates for identified positions on a 0 to 100 number line.

No child correctly identified the numbers 18 and 82 although a considerable proportion did provide estimations to the nearest 10; 19% identified 18 as 20 and 21% identified 82 as 80. Over one third of the children within Year 6 and a quarter of the children with Years 4 and 3 identified 18 as 20. There were only isolated instances of this tendency amongst the children within Years 5 and 2, four children within each year group apart from Year 3 (two children) identified 82 as 80. The tendency for children to identify the nearest ten for both numbers was limited. Four children within Y6 doing so, two within Y5 and Y4 and one within Y2.

7.6 Chapter Summary

This chapter has examined the children's conception of the number line and the ways that children understand it. Thus, the chapter has been a partial response to the question:

> *How is the number line used within the English primary school and how is it understood by children?*

The first part of the question was considered within Chapter 6 and it drew together the way teachers used the number line and the immediate construals that children made of this use. This chapter was concerned with children's general understanding of the number line.

This final aspect has to be inferred from the quality of children's articulations and response to the items within the questionnaire. Table 7.7 indicates the main evidence obtained from the questionnaires and the individual interviews.

Children's embodiment of the number line:	
• **Number line orientation**	Over 80% of the sample prefer number lines that have a left to right orientation The preference is for horizontal lines.
• **Marking an empty number line**	Over 75% of the children appeared to embody the whole of the line as a 0 to 20 line by marking the extreme points.
• **Marking a calibrated line**	Dominant use of whole numbers only
• **Adding additional numbers to a 0 to 5 number line**	70% of children did not respond. Remainder mainly used fractions but half of Year 6 respondents used decimals.
Children's conception of the number line	Largely associated with either descriptions of some features of a number line or some sort of action. Little qualitative difference between the 'definitions' provided by children of different year groups.
Pinpointing numbers on the number line	Accuracy of the estimations decreases as magnitude moves away from 0. Some improvement in accuracy as the numbers move closer to 100. Children exaggerate magnitudes in the first half of the line and underestimate magnitudes in the second half. Distribution of absolute errors across all year groups shows a remarkable similarity although errors decrease as children become older. There is remarkable inconsistency in the difference of the estimates of individual children and the actual numbers. High achievers do not appear to be better at estimating than low achievers.
Estimating the magnitude of pinpointed values	Indications are that there is a tendency to exaggerate numbers in the second half of the line but underestimate numbers within the first half. There are indications that multiples of ten prove to be significant identifiers.

Table 7.7: Children's conceptions of the number line.

It suggests that the children within this sample draw upon a limited perceptual experience of the number line. They associate it with marked extremes, the repeated units identified within terms of whole numbers and the identification of the relevance of the space between the identified whole numbers is confined to a small minority who demonstrate a limited sense of continuity.

Although the National Numeracy Strategy places an emphasis upon children's ability to pinpoint the position of a number on the line, the evidence presents two conflicting outcomes. Free pinpointing evokes either exaggeration or underestimation that would appear to be associated with the perceived position of the number on the line. This does suggest that the children have a mental sense of a line with two halves; the tendency towards increasing accuracy, particularly with the number 55 would also suggest this, since the errors associated with estimating this number are relatively small. Identifying the value of a marked point brings about the reverse trend and also evidence of the use of multiples of ten as key identifying points.

The very limited evidence that the children can add further numbers to numbered and calibrated lines suggest that the sense of the number system of the majority remains framed within whole numbers. There is some evidence that amongst the older children, this sense extends to fraction and decimal, but the evidence also would seem to suggest that this knowledge remains within discrete units rather than part of a global system.

Within the next chapter we attempt to draw together the evidence from this chapter with that of Chapter 6 before presenting the final conclusions.

Chapter 8: Discussion and Conclusion

8.1 Introduction

The question that guided this study:

> *How is the number line used within the English primary school and how is it understood by children?*

has been considered through an examination of its use and understanding by children within one English school. It had not been the original intention that this was to have been the case, rather it was a study that intended to answer the question by following a group of teacher trainees into a variety of schools to give a broader perception of a range of children's understanding and use of the number line. However, circumstances meant that only one school was used. Perhaps this was a good thing, particularly since the school identified was generally perceived to be within the lower half of the achievement spectrum, as measured by the children's attainment within Standard Assessment Tasks, and therefore could be considered as a school which had much to gain from the initiatives presented within the National Numeracy Strategy (NNS) (DfEE, 1999a).

By focussing on the number line, the study may be seen as a means to reflect upon its use as a representation, which on the one hand is seen as a metaphor of the number system (Herbst, 1997) and on the other is recommended as a resource that has strengths as a pedagogic tool (QCA, 1999; QCA, 2000; DfEE, 1999a). The use of the number line as the latter in classrooms (§2.4.3) prompted discussion on its variety of manifestations (Freudenthal, 1973; Anghileri, 2000; Harris and Spooner, 2000), and the possible benefits that may arise from using what has been termed an empty number line (Klein, Beishuizen & Treffers, 1998). In its different manifestations the number line has been used as a classroom resource and as a tool to support not only the development of knowledge and skill associated with the whole number system but also to encourage the reconstruction of knowledge, for example expansion of knowledge of whole number to include knowledge of fractions (Streefland & Heuvel-Panhuizen, 1992; Behr & Post, 1992; Davis, Alston & Maher, 1991), that has the potential to embrace a global perspective of the number system (§2.4.2). To achieve this knowledge there be must an increasing recognition that the line is comprised of an infinite number of points and that each point corresponds to a unique number (Williams & Shuard, 1970). However, this potential may not be realised if there is an over-emphasis on the acquisition of procedural skill (Carr & Katterns, 1984).

Perceptual and conceptual issues that may be derived from the way that the number line is presented to and seen by children (§2.4.4, §2.4.5) may partly be associated with a tendency for teachers to use representations considered to be analogous to the number line in a somewhat ambiguous way (Freudenthal, 1973) without recognising the confusion and limited conceptions that may arise from ignoring their conceptual differences (Skemp, 1989). Children may not appreciate the density concept of the number line, and claim that there are no numbers between two whole numbers, if their experience with the number line has been too strongly associated with 'stepping stones', 'washing lines' or 'number tracks' (Dufour-Janvier, Bednarz & Belanger , 1987). The resulting outcome can be a cognitive conflict as the learner progresses from the discrete nature of natural numbers into the world of rationals and reals (Merenluoto, 2003).

Within the NNS, we see the use of a variety of representations recommended, a long number line, a 'washing line' of numbers, tabletop number lines, marked and unmarked lines and number tracks. Each of these representations are used in a variety of ways to promote understanding of the sequence and order of the naturals, introduce and develop addition and subtraction, and expand children's knowledge of the number system to include fractions, decimals and negative numbers. However, within the document, there is no clear indication of the conceptual differences between each of these representations and neither is there explicit treatment of what it may mean to possess a relational understanding of the number system. Illustrations may provide a sense of the implicit structure of a number line but it has to be assumed that the depth of structural interpretation and implications for consequent use is left to teachers. A consequence of this is, of course, that the way the teacher presents the knowledge influences the way in which the child will understand it. Teachers, as well as children, form personal constructions of knowledge (§4.2.1) and it is from this perspective that one conclusion that can be drawn from this study is that the teacher's constructions are based upon the recommendations they see within the NNS. From the evidence of the teacher observations within this study these recommendations would not only appear to play a large part in the formation of the teacher's subject matter knowledge (§2.5.5), but would also appear to influence their style of teaching to the extent that it is largely based upon transmission (Askew, Brown, Rhodes, Wiliam & Johnson, 1997c) with associated number line considerations presented in demonstration mode (Foster, 2001).

Herbst (1997) suggested that it is the structural features that arise from the definition of the number line (§2.4.2) that permit its use as a pedagogic tool. It has been the influence or otherwise of the relationship established between these structural features and the pedagogic use of the number line that has formed the basis for the evidence collected for this study. Teacher perceptions of the nature of the number line and indications of the way they suggest they use it in the classroom (Chapter 5) was followed by observations of their actual use and children's construction established from this use (Chapter 6). The children's understanding was considered within Chapter 7.

The purpose of this chapter is to draw these elements together. It does this initially by considering the relationship between teaching and learning (§8.2) and then it considers children's understanding of the structure of the number line and their ability to estimate magnitudes on a number line segment (§8.3). The discussion within both of these sections is associated with the teaching guidelines and outcomes specified within the NNS. Section 8.4 presents reflection on the method used within the study whilst section 8.5 considers additional research that may arise out of the study. A final reflection is presented within Section 8.6

8.2 Teacher Understanding and Use of the Number Line

The Framework within the National Numeracy Strategy "illustrates the intended range and balance of work in primary mathematics to ensure that pupils become properly numerate" (DfEE, 1999a, p.2). Numeracy is identified as a proficiency in numbers and measurement and, amongst other things, it requires an understanding of the number system and a repertoire of computational skills. To achieve this end, two of the five strands of development identified within the NNS are 'Numbers and the Number System' and 'Calculations'. Having direct links to the National Curriculum Programmes both of these strands have illustrative examples that use the number line as one resource that would support the attainment of yearly outcomes. Appendix I illustrates how the number line is presented as a representation that may support the achievement of objectives associated with children's whole number development from numbers to 10 (Reception) to numbers to 10000 (Year 5) with a complimentary emphasis on development of fractions (from Year 3), Decimals (from Year 4) and integers (from Year 5). The numerical knowledge and understanding at the end of Year 6 should include knowledge of any number associated with the earlier development.

The examples within the NNS that are associated with the number line contain several recurring themes, the most dominant being those associated with order and position, estimation and counting (see Appendix II). Frequently, we see outcomes associated with the use of the number line in dispersed with outcomes associated with the use of the number track and the hundred square (see Appendix II, Year 2, Year 3 and DfEE, 1999a; Sections 3.10.15, 3.14.15, 5.8, 5.15).

Given that the initial training programme of the teacher trainees and the classroom practices of the teachers have been somewhat immersed in the guidelines presented within the NNS, it is perhaps not unsurprising that almost 50% of the former indicate that the number line is a representation that is associated with ordering numbers but few of these mention that the numbers could be other than whole numbers (§5.2.1). To define a number line, the teacher trainees appeared to rely on description of a specific line limited to the more obvious perceptual characteristics rather than the underlying conceptual features. Approximately one quarter of the students made reference to equal divisions between the numbers, but none indicated that these divisions were formed as a result of successive repetition of the first interval — the unit interval. One in ten of the teacher trainees associated the number line with some sort of action, and in particular its use in counting.

Three of the five classroom teachers interviewed, also associated the number line with the ordering of numbers whilst all considered it a useful resource in developing children's numerical skill. However, perhaps the most significant feature arising from informal interviews with teachers was the way within which they appeared to see the number line, the number track and the hundred square as almost the same thing, but none made an explicit structural distinction between them as representation, an omission that, as we have seen, is also apparent within the NNS guidelines. The consequence was that the defining features of the number line and the number track were absent from discussion within the classroom.

The NNS guidelines suggest that a high proportion of lesson time should be devoted to direct teaching of whole classes (and groups) during which teachers should demonstrate, explain and illustrate mathematical ideas by, amongst other things, giving children access to number lines to model mathematical ideas and methods. When teachers were asked in interviews to indicate when and how they use the number line (§5.3.2), their responses focused on ordering of numbers, using it as a tool for the performance of arithmetic operations (i.e. counting forwards and backwards, addition or subtraction) but they only made implicit rather than explicit references to its use in expanding understanding of the number system.

The use of the number line within the observed lessons was generally initiated through demonstration and explanation to introduce and explain ideas such as addition and subtraction and then eventually processes such as bridging ten or using the multiples of ten. Foster (2001) had suggested that using a representation to demonstrate ideas — demonstration mode (§2.3.2) — was a means through which teacher and student could communicate ideas about the performance of procedures rather than a means through which a sense of the actual representation could be transmitted. Within the observed lessons the former was much in evidence and in some instances, as in the case of carrying out a subtraction procedure, repeated across several year groups, "Just put the largest number first" (Y2 teacher), "The largest number has to come first on a subtraction" (Y3 teacher), "Smallest (number goes in the beginning) coz the number line goes up" (Y4 teacher). Subsequent actions were then associated with a "jump" from the first placed number to a second.

The number lines used by the teachers were always presented horizontally and if calibrated, possessed a left to right orientation, started most frequently at zero and, apart from the exceptions when lessons were specifically designed to consider fractions, decimals or the integers, consisted of whole numbers. The discourse associated with the use of the line appeared to be common across all of the classes. Thus, there appeared to be a standard representation, one that was used by all teachers, which, it is conjectured, through the variety of interactions, they had with such models, served to provide the basis for the children's embodiments of the number line (Lakoff & Johnson, 1999).

However, within the age related lessons observed, as children became older, it appeared to be implicit in the teacher's use of the number line as a tool that the 'standard' representation was in an embodiment shared by all children. Initial references to the number line that had also included ambiguous references to the number track and the hundred square (for example §6.2.2) became minimalist in that first an unmarked line became the source of demonstration (for example §6.2.3) until eventually it was jumps alone that were represented (§6.2.6). These changes were associated with complete operational use of the number line to count, add and subtract, the transformation to note 'jottings' to record a process, to the representation of whole number partitioning accompanied by the need to identify remainders as fractions. The general approach to recording fractions or decimals appeared to place the intervals under a higher degree of magnification so that they could be partitioned.

The relationship between these shifts from calibrated number line segment, to an open number line segment (with an interval initially identified or identified to support the application of a procedure) to the magnification of a unit or succession of unit intervals was not made obvious to the children during any of the observations. Thus, opportunities to enhance the children's conceptual understanding of the line were not taken. Instead, these shifts appeared to be associated with procedural outcomes specified within the NNS under, it is conjectured, the teacher's assumption that the children's embodiments of the number line were taken as shared with their own.

Cobb, Yackel & Wood's (1992) suggestion that the dualism associated with such an assumption may be associated with misconceptions is clearly seen in the examples drawn from the Y4 children's experience in fractions (§6.3) and in the ways that some of the children within Y4 (§6.2.4) and Y5 (§6.2.5) attempted to use a number line for addition.

Given the ways the number line was presented within the classrooms, particularly within Years 2, 3 and 4, it was not unsurprising that the children's embodiments of the number line were associated with a horizontal or slightly diagonal line with left to right number orientation (§7.2.1). By far the greater majority of the children (93%) identified the range of a number line by marking the two ends (§7.2.2.1) and only 7% (1 child from each of Y3, Y4 and Y5 and 2 from Y6) who, though they marked the left extreme 0, chose to provide the second reference number (20) at some other point on the line.

Recognition that the unit intervals of a number line could be partitioned so that points identifying fractions or decimals could be noted, was confined to approximately one quarter of the children from each of the Years 4, 5 and 6 (§7.2.2.3) but only child from Year 6 included decimals, although an NNS outcome for Year 4 indicates that children should be taught to place decimals on a number line segment. Almost 60% of the children who were invited to add additional numbers to a 0 to 5 number line segment with the whole numbers marked, did not respond. An NNS outcome for Year 2 children suggests that the children should begin to position halves on a 0 to 10 number line segment, whilst the example identified within the NNS (1999a, Section 5, p.23) was the exact example taught by the Y4 teacher (§6.3). The conceptual limitations suggested by the children's non-response to the addition of additional numbers was perhaps reflected in the fact that when asked to place numbers on a calibrated line, 97% of the children chose to place whole numbers (§7.2.2.2).

In their broader perceptions of what a number line was (§7.3), the children either simply described some perceptual features of a particular line or explained a particular line in the context of an action. Overall, the quality of the children's responses did not change significantly between those given by Year 3 children and those given by Year 6 children. This feature, together with their over-riding 'preference' to label calibrated lines with whole numbers and the limited acknowledgement that an interval could be partitioned, suggests that the children's embodiment of the number line was formed from their relatively early experiences and changed very little with subsequent experience. Thus, by the time teachers are using a number line to demonstrate the partitioning of intervals for fractions and decimals, for many of the children, the notion of 'number line' carries an embodiment pre-loaded through prior active, linguistic and relational experience with whole number (§2.5.5) leading to the sort of outcomes noted by other studies that consider the number line in, for example, the context of fractions (§2.4.5). Without a sense of the conceptual structure that underscores the use of the number line, the notion that it is an ideal representation for connecting whole number and fraction (Baturo & Cooper, 1999), an aspect of use frequently seen within the NNS, must be questioned.

8.3 Children's Estimation of Magnitude

Estimating magnitudes is a recurring activity associated with desirable learning outcomes specified within the NNS (DfEE, Section 5; pp. 8, 9, 17; 1999) and it was an activity frequently seen during the Y3, Y4 and Y6 lesson observations (§6.2.3, §6.3, §6.4). Nevertheless, when children were asked to pinpoint numbers on a 0 to 100 number line segment the trends associated with their errors suggested that whether or not a child was a high achiever or a low achiever was not an indicator of accuracy, that children did not necessarily reflect consistency in their degree of accuracy but, in general, older children were more accurate than younger children.

However, there appeared to be a general pattern in the distribution of errors, which suggested that magnitudes in the first half of the line were exaggerated whilst those in the second half were underestimated.

Dehaene (1997) suggested left to right orientation, where smaller numbers are associated with the left hand side of space and larger numbers tend to be compressed into a smaller space associated with the right hand side, is an inherent and intuitive mental idea of the number line we all posses, which he calls "number sense". He further suggested that on this mental number line not all numbers are represented with the same accuracy.

> Subjectively speaking, the distance between 8 and 9 is not identical to that between 1 and 2. The 'mental ruler' with which we measure numbers is not graduated with regularly spaced marks. It tends to compress larger numbers into a smaller space. ... As soon as a continuum needs to be divided into discrete categories, intuition dictates the selection of a compressed scale, most often logarithmic, which tightly matches our internal representation of numbers. (pp. 76, 77)

In Galton's (1907) data this left to right orientation was also dominant, and the associated imagery included distinct images for smaller numbers and key numbers such as 12 (derived from the association with money — 12 pennies equal one shilling??) which suggest the mental embodiments of a "guide point". Dehaene suggested that the error for smaller numbers is smaller because humans express smaller numbers more often than larger numbers throughout their lifetime. Smaller numbers are taught first and they are repeated consistently, and the evidence from the current study suggests that this is often associated with number line segments from 0 to 10 or 0 to 20.

The work of Siegler and his colleagues into the ability of children to estimate magnitudes became apparent late in the development of this study and it possessed a remarkable similarity to the investigation carried out in §7.4 in that it focused on number line estimation tasks with children within the 2nd, 4th and 6th grades.

Siegler & Opfer (2003) and Siegler & Booth (2004) argue that estimations associated with the representation of numerical quantities on a number line can be either logarithmic or linear or an amalgam of the two. Linear representations suggest a constant growth in the magnitude of the errors and are the result of an approach to estimation where the individual divides the line at specific points and uses these points as references to locate other numbers. A logarithmic scale would suggest that as we move away from zero errors associated with estimations become larger. Thus from the perceptual perspective they would seem to concur with Dehaene's conjecture within the perspective of mental imagery.

Siegler & Opfer (2003) concluded that the 2^{nd}, 4^{th} and 6^{th} graders estimations of numbers in the range 0 to 100 expressed linear mapping between the numbers and their magnitudes that improves in consistency with age. The children's estimation of numbers from 0 to 1000 expressed a logarithmic pattern which:

> ... exaggerates the distance between the magnitudes of numbers at the low end of the range and minimizes the distance between magnitudes of numbers in the middle and upper ends of the range. (Siegler & Booth, 2004; p. 429)

Siegler & Booth (2004) concluded that kindergarteners' estimations expressed a consistently logarithmic pattern, 1st graders an amalgam of logarithmic and linear patterns and 2nd graders purely linear. Siegler (2005) suggested that estimations in general are far from accurate because of limited real knowledge and conceptions of numbers. He also suggested that linear representations are a result of age and experience, are associated with high achievement and are the result of an approach to estimation where the individual divides the line at specific points and uses these points as references to locate other numbers.

Certainly the evidence from this study drawn from children within one school does not confirm that the accuracy of estimations is associated with achievement (§7.4.2, Figure 7.9) although some caution must be applied to this statement given the general nature of the achievement within the school. Thus, we do not use the notion of "high achievement" but only make the observation in the context of the children who formed the basis for the sample. In addition, however, the distribution of the means of the absolute errors associated with each year group attempting to estimate particular numbers, does not appear to have either a linear nor a logarithmic scale although there are indications that perceptual reference points may guide some of the estimations.

Figure 8.1, is constructed in the same way as Figure 7.12 (§7.4.3.1) but is a diagrammatic illustration of the distribution of the means of the absolute errors associated with each year's attempts to estimate. Whilst it illustrates the patterns of the error distributions by indicating their size and direction, it also serves a comparative purpose by establishing five different lines for each of the Years 2 to 6. Vertical lines indicate the magnitude of the numbers children had to pinpoint and reference to the 'Difference' scale indicates the size of the mean absolute error associate with each number estimated. To achieve simplicity the figure does not include pinpointed differences.

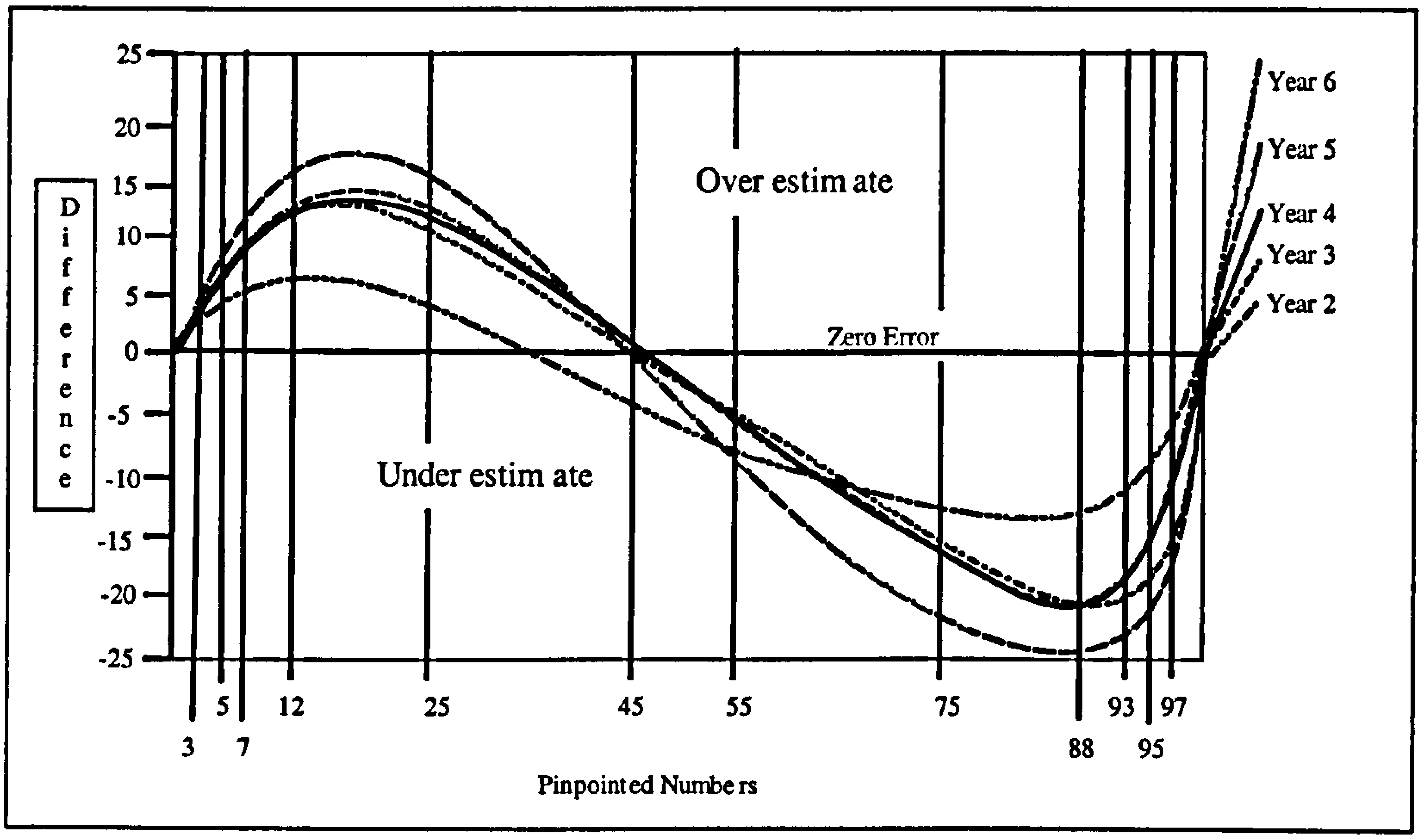


Figure 8.1: Summary of general trends associated with errors in children's efforts to estimate position of numbers on a number line

What is striking from Figure 8.1 is that the distribution of the means of each year group should have such a remarkable similarity — the children exaggerate magnitudes in the lower half of the line but underestimate them in the upper half.

Additionally, it can be seen that:

- the maximum extent of underestimation can be greater than that of overestimation.
- the trends noted amongst children within Years 3, 4 and 5 are similar. Years 2 and 6 demonstrate departures from these in that differences identified within Year 2 are greater than those of all other years, whilst those identified for Year 6 are less.

- the remarkable accuracy with which all year groups, apart from Year 6, appear to identify the position of 45, but this is considered in more detail later since it is the only number where the children differ in their tendency to under-estimate or over-estimate.

Given that two points marked on the number line segment were 0 and 100 it is perhaps not surprising that as children are required to pinpoint numbers close to the points, the errors that arise are generally small and unidirectional (Table 7.6). The accuracy of estimation would appear to increase the nearer the required estimation is to an end point. Near the end points, there is a decreasing opportunity to estimate to an alternative side of the target — to make a negative error when pinpointing near zero, and to make a positive error when pinpointing near 100.

This evidence would also seem to support the conclusions of Siegler and Booth (2004) in that the children exaggerated magnitudes in the lower half of the line, but underestimate in the upper half. However, the nature of the distribution of the differences does not support either a linear or a logarithmic theory. Nor does it allow us to subscribe to the view that with the range of children considered there is a change in the pattern of the distribution — although there is some evidence of growing accuracy, this appeared to stagnate between Years 3 and 5.

We conjecture that the crests in Figure 8.1 are due to the children's tendency to use counting as a procedure to provide their estimates — a procedure observed throughout the spectrum of primary school mathematics in many of the number line activities observed both in teacher demonstration and in children's actions (see Chapter 6) —and the troughs are due to the influence of "visual correctors" such as 0, 100 and 50. It is conjectured that experience associated with finding the middle of a number line segment (§6.2.3, §6.3) may have contributed to the relative accuracy with which 45 and 55 were estimated by children from within Years 3, 4 and 5. It was the range of the estimates (29 to 55 for 45, for example, with the average estimate being 40.2) identified amongst some of the children within Year 6 and the consistent tendency to underestimate (16 of the sample of 19) that meant that the mean of the absolute errors for these numbers was, in comparison to the other years, relatively less accurate. Thus, it is conjectured that for many of the children, because of their classroom experience, the middle of the 0 to 100 segment, together with their number sense that this was 50, enabled them to place greater accuracy on the estimates near the middle of the line. However, given the size of errors associated with the numbers 25 and 75 evidence suggesting an overall tendency to divide the line at specific points and use these points as references to locate other numbers (Siegler, 2005) is not apparent from the size of the errors associated with each year group. It is further conjectured that since almost 60% of the full sample left evidence of dots, small lines or even numbers, all starting from zero, on the sheets within which they responded to the estimates, counting procedures dominated their estimation strategies.

However, it may be that a clearer view of the children's behaviour may be related to Fischer's (1994) suggestion that the bias, towards accurate estimation, is associated with perception, both real and imagined and that in some instances perception overcomes procedural approaches. Such a view would go someway into accounting for the smaller errors, and therefore the greater degree of overall accuracy, associated not only with 45, but also with 3, 5, 95 and 97 relative to other numbers in their half of the number line segment. Whatever the real reasons, the results would suggest that the experiences advocated within the NNS, and evidenced through the observations, particularly in the early years, do not appear to readily transfer.

The findings of the study have implications for the learning expectations and the teaching approaches suggested within the NNS. The evidence confirms and complement other research findings on the use and impact of pedagogic representations for the learning of mathematics (§2.4.4, §2.4.5). The NNS does not indicate that the number line should be presented and developed as an abstract conception of the number system, but it is presented as a concrete model that supports actions. Whether or not the authors of the NNS made an assumption that the conceptual issues associated with the former may be raised as children deal with the varying aspects of the latter, the evidence from teachers' perceptions of the number line suggests that making this link is an issue associated with teachers' subject knowledge. This would appear to be relatively superficial but if addressed, would improve their pedagogical content knowledge and the emphasis they place on the use of the number line within the classroom.

Bright, Behr, Post & Wachsmuth (1988) and Behr & Post (1992) suggest that while the number line is generally an effective model, it is also the source of difficulty both in instruction and its use by children. They indicate that the number line can help children learn about addition and subtraction and that it can strengthen children's understanding of fraction order and equivalence, as long as it is not the first model used. This latter point is supported by Dufour-Janvier, Bednarz & Belanger (1987) who stress the importance of the use of a representation at the right time:

> The premature use of a representation, as well its application in an inappropriate context, can lead children to develop misconceptions that will hinder them in later learning.
>
> (p. 117)

There is something of a contradiction between these recommendations and the evidence obtained from the observations of the Year 1 and 2 classes where there is extensive use of the number line (§6.2.1, §6.2.2). The recommended conceptual understanding of the representation suggested by Herbst (1997) is not considered and therefore use of the number line as a pedagogic representation for procedures takes place without the structural basis, which could have provided a meaningful understanding of the representation and, particularly in later years, a more successful appreciation of its use.

Despite these results and the evidence presented within this study, the 2004 Ofsted inspection report indicated that the mathematics curriculum had been implemented successfully and that it meets pupils' needs satisfactorily, enabling all groups of pupils to achieve well within the participating school:

> The National Literacy and the Numeracy Strategies have been fully and successfully implemented. These have been the main foci of teaching over the past four years.
>
> (OfSTED report 2004, p. 13)

It appeared to concur with the school's intention to shift the emphasis away from literacy and numeracy because:

> In recent years there has been a high focus on literacy and numeracy and the school has now identified the need to improve the provision for, and the links between, the non-core subjects. (OfSTED report 2004, p. 7)

8.4 Reflections on the Methodology - Limitations of the Study

Inspired by evidence from the teacher trainees within the exploratory study (§3.2), the initial intention of the main study was to relate the teaching and learning of the number line with a selected sample of final year teacher trainees in the UK (§4.4.1). Following an initial appraisal of their understanding of the number line, this was to be followed by a six-month period observing the lessons where the teacher trainees used the number line and in gaining a perception of their classes interpretation of this use and of their perceptions of the number line. By carrying out a study of this type, it was intended to gain evidence not only about children's understanding of the number line, but the way that teachers emphasised it within the classroom and what children made of this emphasis.

Nevertheless, because of practical difficulties (see §4.4.1), the main study had to be modified. For an investigation on teaching and learning, free access to pupils as well as classrooms when the number line was used was required. For this reason, one particular school was used and consequently, the practicing teachers within this school participated in the study.

The purpose in establishing the samples was to:

(a) Observe teaching and learning associated with the number line

(b) Investigate children's understanding of the number line

The decision to attempt to observe across the spectrum of a school had strengths and weakness.

Its strength was that it gave an indication of how a range of teachers used the number line and therefore from the action perspective allowed the researcher to consider the children's understanding as they expanded their experiences of using the number line from whole number to fractions and decimals. Of particular interest was the relationship between the conceptual strengths of the line and their relationship with its use as a tool. Its weakness was that the researcher was reliant upon teachers to inform her of when the number line was to be used. This generally worked well within the school, but it also meant that the researcher was unable to see those instances when the number line may have been used as an incidental pedagogic aid in teaching.

An additional problem was the reluctance of the teachers to give the researcher an extended opportunity to discuss their use and understanding of the number line; partly because of the pressures that they felt on their time, for example lunch times were very short and most used the time to prepare for afternoon sessions whilst they also had meetings of one form or another immediately after school. Any comments from teachers were generally opportunistic and whilst they had value, it is recognised that this value is somewhat limited. For this reason, data obtained previously on teacher trainees' understanding of subject knowledge (from a questionnaire) complemented the practicing teachers' data.

It all brings into question the ability to carry out this form of research within school unless the researcher is actually part of the school or unless people volunteer for the research to be part of a broader research programme that is funded to allow teachers' time to participate. In this particular case, although volunteers were initially identified — student teachers — the practicalities of following these students in any meaningful way could not be easily overcome, so although problems remained it is with gratitude that I thank the school and the teachers who agreed to participate as an alternative.

8.5 Proposals for Future Research

Considering the issues that form the main component of this chapter, there would appear to be a need to gain greater insight into the ways that teachers understand the nature of the representations they use. The evidence from this study suggests that teachers think of the number line in terms of its use as a tool associated with some form of action, but they do not appear to have a personal understanding that includes its nature as a sophisticated representation of the number system. Their conscientious efforts to develop children's knowledge through demonstrating procedures and then the provision of opportunities to rehearse these procedures may only serve to limit children's understanding of the representation used to develop the procedure.

While this study has provided enough evidence from teachers to back up the arguments, there could always be additional data adding to the teacher's personal understanding of the number line in greater depth. The results of the study indicate that despite the guidelines within the National Numeracy Strategy, the way instruction takes place does not really contribute towards the development of conceptual understanding and the construction of meaningful learning in the use of the number line by children within this particular school. Given that the school appears to be generally below average in the levels of attainment achieved by the children, and this is not because of disciplinary nor management defects within the school, indeed, quite the reverse, it is a well ordered school that does its best for children from difficult home environments, the purpose of the NNS strategy as a programme that is seen to remedy deficiencies in knowledge and skill for these particular children, and possibly for many like them, should be reconsidered. As a programme that sets outcomes to be achieved each year, its value must be questioned. A deeper understanding of the somewhat bland relationship between content and the use of resources to develop content requires even more study that it has had to date. Associated with this should be a closer study into the relationship between the primary school teacher's subject knowledge and their pedagogic practice. Even though the mathematics may seem to be elementary from the content perspective and the conceptual structures of the resources used, self-evident to those that know them, no assumptions should be made. Thus, studies into the relationship between teacher subject knowledge, pedagogy and learning should continue to be a high priority.

This study carries an implication that conscientious teachers have attempted to meet two requirements; that of ensuring their students understand mathematics and that they can also do mathematics in a way that would enable their children to satisfy the external requirements. These two aims are not incompatible but neither are they necessarily compatible but the quality of the relationship between the two in the context of curricula designed to improve understanding and yet influenced extensively by the needs of assessment does require reflective consideration.

8.6 Final Thoughts

The conceptual differences between the number track (and the hundred square) and the number line are not a cause of concern for the teachers that have been observed —both can be used to illustrate "jumps". The fact for the number line the jump is from an end point of an interval to the end point of an another interval whilst on the number track the jumps exist between spaces does not appear to be a matter of concern — it is the numeral and the difference between it and another numeral that is focused upon and not upon what the numerals represent. Both can equally be used as perceptual representations that support counting on or counting back in ones or in sequences of ten. The teachers focus on them as representations that they are encouraged to support the development on their children's skill and understanding in numeracy. However, whilst they make the former explicit, the latter requires a process of reflective abstraction.

In choosing the children for individual interviews, by considering the frames of reference they applied to concrete and abstract objects they had experienced, it was hoped that such abstraction would be evident in the children's articulations during interview. This was not, apart from one exception (Child 6.2), the case. All of the children articulated their understanding and use of the number line in a way that emphasised episodes and specific representation. The quality of abstraction portrayed by the children emphasised actions and specificity. There was little evidence to suggest that the children drew more than simply the actions from their experience. They did not appear to draw everything from these actions, for example the notions of end point to end point, reconstruct them at a higher level (§2.2.1) to recognise the structures underpinning the number line.

To reach this level, children need guidance but this guidance was absent from the document that was the key curriculum resource and from the teaching they experienced. Consequently, most of the children that the document and the teaching was designed to support did not make the appropriate conceptual leaps. The interpretations they made of the number line, the actions associated with it and the communication of mathematical meanings did not give them a global picture of a number system but instead a picture made up of several discrete packages, whole numbers, fractions and decimals. As each package developed, their understanding of the number line remained embodied in a whole number world that, it is conjectured, did not reflect the assumptions of the teachers. The sophistication that may be gained from recognising the intuitive completeness of the number line did not appear to be achieved by these children and therefore it did not appear to contribute towards their full development of understanding of the number system.

This all leads to the thesis of this study:

> *The primary use of the number line as a tool can undermine its strengths as a metaphor of the number system and may disadvantage children in their continued reconstruction of knowledge of that system.*

It is a corollary of this thesis that use of the number line, as a representation within the primary school should be withheld until such time as children understand its conceptual structure.

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Appendix I

Objectives within the NNS: An Overview of Number Development

Year	Knowledge	Skill	Representation
Reception	Number to 10 (2.2)	Count, recognise, compare and order numbers. Relate addition and subtraction to combining and "taking-away" groups of objects.	Estimate a number in the range that can be counted reliably, then check by counting. (3.2.8) Use a number track point to different numbers and say what they are. (4.8) Begin to write numerals correctly, tracing from top to bottom in a continuous line where possible, first 1,2,3... then 0 to 5, progressing to at least 10. (4.10)
Y1 (KS 1)	Number to 20 (2.2)	Count, read, write, compare and order numbers. Understand the operations of addition and subtraction.	Order numbers to at least 20 and position them on a number track. (3.6.14) Write numerals on a blank number line. (5.8)
Y2 (KS 2)	Number to 100 (2.3)	Count, read, write, order, round and compare whole numbers. Understand that subtraction is the inverse of addition. Estimate, measure and compare lengths.	Order numbers to at least 100 and position them on a number line and 100 square. (3.10.15) Record estimates on a number line and find the difference between the estimate and the actual number. (5.17)
Y3 (KS 2)	Number to at least 1000 (2.3)	Read, write, order, round and compare whole numbers. Add and subtract mentally. Recognise fractions 1/2, 1/3, 1/4, 1/5, 1/10. Estimate a simple fraction.	Order numbers and position them on a number line and on a number track. (3.14.15) Compare familiar fractions: for example, know that on the number line one half lies between one quarter and three quarters. (3.14.21,23) Record estimates on a number line and find the difference between the estimate and the actual number. (5.17)
Y4 (KS 2)	Number to 1000 (2.4)	Read, write, round numbers. Recognise simple fractions as parts of a whole and equivalent fractions. Order simple fractions. Recognise the equivalence between the decimal and fraction forms of one half and one quarter, and tenths such as 0.3. Carry out column addition and subtraction.	Recognise negative numbers in context (e.g. on a number line, on a temperature scale). (3.18.14)
Y5 (KS 2)	Number to 10000 (2.4)	Read, write, compare and order numbers. Use decimal notation for tenths and hundreds, round decimals to the nearest integer. Relate fractions to division	Order a given set of positive and negative integers (e.g. on number line, on temperature scale). (3.22.15) Order a set of fractions such as 2, 2 3/4, 1 3/4, 2 1/2, 1 1/2, and position them on a number line. (3.22.23)
Y6 (KS 2)	Any Number (2.5)	Order a mixed set of numbers with up to three decimal places. Find fractions of numbers or quantities. Carry out column addition of numbers involving decimals.	Find the difference between a positive and a negative integer, or two negative integers, in a context such as temperature or the number line, and order a set of positive and negative integers. (3.26.15) Order fractions such as 2/3, 3/4 and 5/6 by converting them to fractions with a common denominator, and position them on a number line. (3.26.23)

Appendix II

Examples Provided within the NNS

NNS Reference	Provided Examples	Objective	Category
Recep (4.5)	Count objects in a line: first touching them one by one, then without touching them.	Count systematically to keep a track of the count	Counting & Ordering
Recep (4.9)	Look at this number track. Count along with me. Say each number as I point to it. Who can point to 7 on the number track?	Recognise & use numerals 1 to 9, extend to 0 & 10 then beyond 10	Interpolation - arrow
Recep (4.12.a)	Here is part of a number track. Can you say what the missing numbers are? (1,...,3,4,...,6,7,...,9,...)	Say a number lying between two given numbers	Missing numbers
Recep (4.12.b)	Which two numbers have been changed over on this track? (1,2,3,4,8,6,7,5,9,10)	Order a given set of numbers	Counting & Ordering
Recep (4.12.c)	Put in order, smaller first, a set of numbered carpet tiles 1 to 10, with three or four of the numbers removed. Which numbers are missing? (in boxes: 5,7,9,10,4,2)	Order a given set of selected numbers	Missing numbers
Recep (4.15)	Make a hop of three spaces on the number track. Now hop two more. Where are you?	Begin to relate addition to counting on	Operations
Y1 (5.2.a)	What number comes after 6? After 17? Before 9? Before 14?	Know & recite number names in order from and back to zero	Counting & Ordering
Y1 (5.2 b)	Here is part of a number track. Where does 9 go? And 2? (with only 4,5,6 in middle boxes)	Counting in ones	Missing numbers
Y1 (5.10)	Write the numbers between 3 and 9 on the number track (1,2,3,...,...,...,...,...,9,10)	Compare & order	Missing numbers
Y1 (5.12)	Fill in the missing numbers on this number track (...,2,3,4,...,6,...,8,9,10,...,12,13,...,15)	Order & position	Missing numbers
Y2 (5.3.a)	Start at any two-digit number and count on in ones to 100, or back in ones to zero.	Counting in ones	Counting & Ordering
Y2 (5.3.b)	Here is part of a number track. Where would 42 be? 33? (with 36,37,38,39 in the middle)	Counting in ones	Missing numbers
Y2 (5.11)	What number is half way between 10 and 20? (number line with notches for tens - arrow in middle)	Estimate	Interpolation - arrow
Y2 (5.15.a)	Fill in the missing numbers on this number line. (line with boxes under it: ...,69,...,71,72)	Order & position	Missing numbers
Y2 (5.15.b)	This is a 0 to 100 line marked in tens. Write where these numbers go on the line: 20, 60, 90...	Order & position	Interpolation – no arrow
Y2 (5.17)	Estimate the position of a point on a line. For example, estimate the whole number marked by the arrow. How did you decide? (line 0 to 10)	Estimate	Interpolation – arrow
Y2 (5.23)	Begin to position halves on a number line. For example, place 5 ½ on a number line, and recognise that it lies mid-way between 5 and 6. (number line with whole numbers 0 to 10)	Recognise & find simple fractions	Interpolation – no arrow
Y3 (5.3)	Here is part of a number track. Where would 142 be? 132? (with 136,137,138,139 in the middle)	Counting in ones.	Missing numbers
Y3 (5.11)	What number is half way between 40 and 60? (number line with notches fro tens – arrow in middle)	Estimate	Interpolation – arrow
Y3 (5.15.a)	Fill in the missing numbers on this number line. (line with boxes under it: ...,256,257,...,259,...)	Order & position	Missing numbers
Y3 (5.15.b)	This is a 0 to 100 line marked in tens. Mark where these numbers go on the line: 28, 65, 92...	Order & position	Interpolation – no arrow
Y3 (5.17)	Estimate the position of a point on a line. For example, estimate the whole number marked by the arrow. How did you decide? (number line 0 to 100)	Estimate	Interpolation - arrow
Y3 (5.43)	Counting on in multiples of 100, 10 or 1 86+57=86+50+7=136+7=143 (use empty number line with jumps 50,4,3) 356+427=356+(400+20+7) (use empty number line with jumps 400,20,4,3)	Pencil & paper methods for additions that, at this stage, cannot be done mentally	Operations
Y3 (5.45)	Counting up from the smaller to the larger number (complementary addition) 86-56=56+4+20+4=84 (use empty number line with jumps 4,20 4)	Pencil & paper methods for subtractions that, at this stage, cannot be done mentally	Operations
Y4 (5.8)	This is part of the number line. Fill in the missing numbers. (line with hanging boxes ...,3298,3299,...,...,...,...)	Compare & order	Missing numbers
Y4 (6.8)	Indicate on a number line what number is half way between 740 and 750, 4000 and 4100. (number line with notches from tens – arrow in middle)	Compare & order	Interpolation – arrow
Y4 (6.10)	Estimate the position of a point on an undivided line: for example, the whole number marked by the arrow. Explain how you made your decision. (ENL from 0 to 100 with an arrow)	Estimate	Interpolation – arrow
Y4 (6.14.a)	Fill in the missing numbers on this part of the number line (from –6 to 2 in ones and –4 to 4 in ones).	Recognise & order	Missing numbers
Y4 (6.14 b)	Draw an arrow to point to –2 (number line from –4 to 4 with notches for the whole numbers)	Recognise & order	Interpolation – no arrow
Y4 (6.28 a)	Count along this line and back again (a line from 0 to 1 and notches for the tenths).	Order decimals	Counting & Ordering
Y4 (6.28.b)	Place these decimals on a line from 0 to 2: 0.3, 0.1, 0.9, 0.5, 1.2, 1.9 (line with marks for tenths and 0,1,2 on it)	Order decimals	Interpolation – no arrow
Y5 (6.11)	Estimate the position of a point on an undivided line: for example, the whole number marked by the arrow. Explain how you made your decision. (ENL from 0 to 1000 or –5 to 0 with arrow)	Estimate	Interpolation - arrow

Appendix III

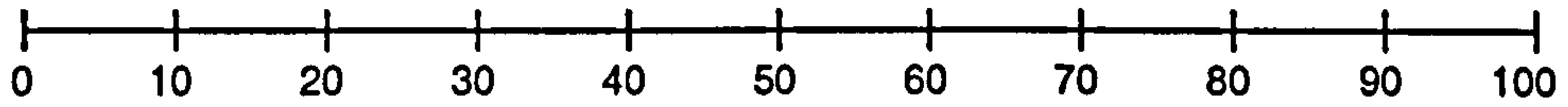
Preliminary Studies Questions

Questions

1) What comes to your mind when you think of the word 'number'?

2) What comes to your mind when you think of the numbers from 1 to 100?

3) Do you know what this is? (wait for response) Do you know what a number line is? Could you point at number 40 for me? What about 79? And 3?

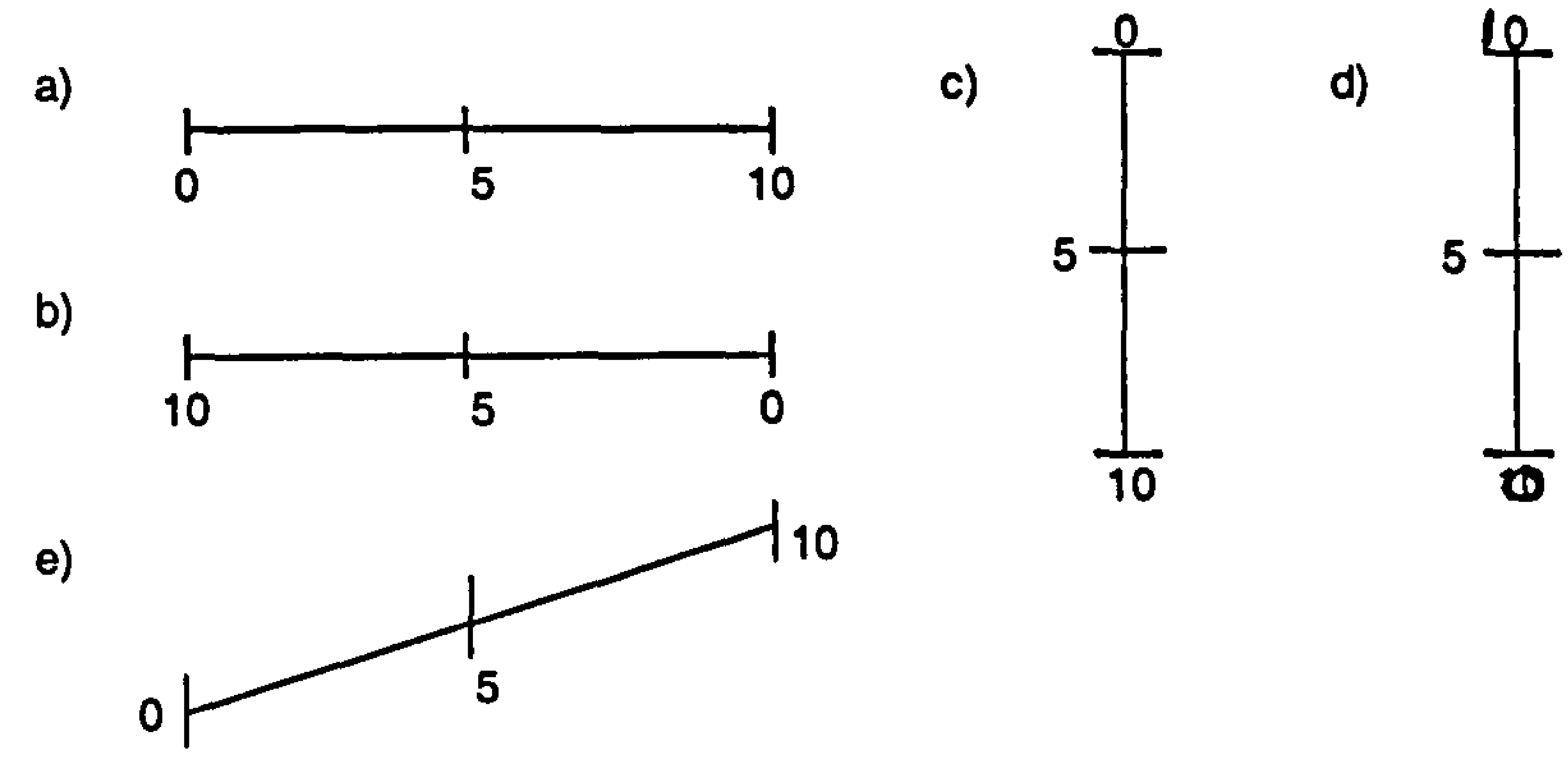


4) Where do you think the following numbers would go on this line? Mark the point. (The numbers were: 93, 45, 12, 5, 75, 3, 7, 25, 97, 95, 88, 55)

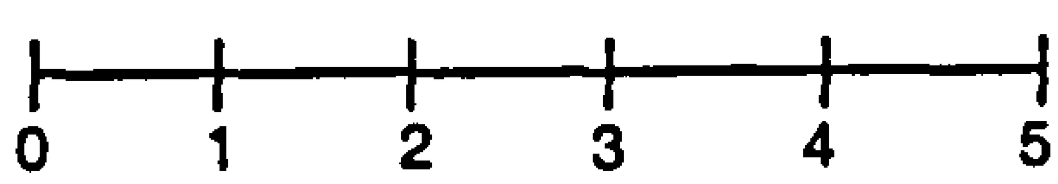
Appendix IV

Main Study Questionnaire

1) Here are some number lines. Tick the one you like the most. Explain why



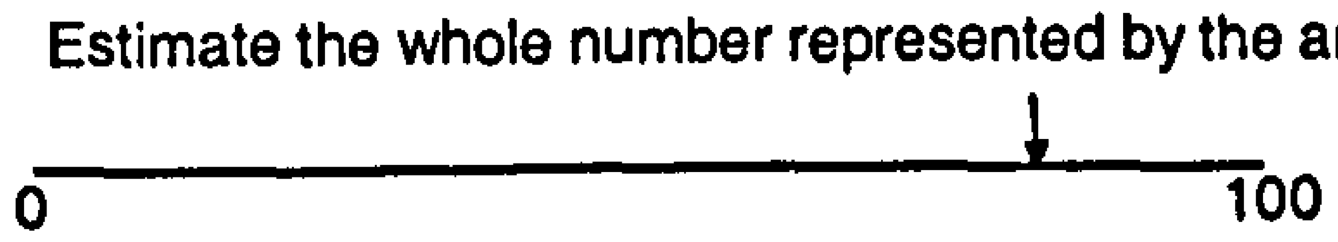
2) Can you put more numbers on the line?



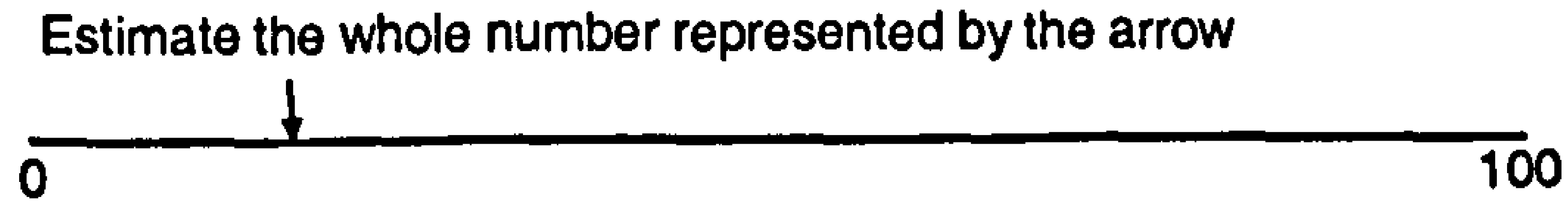
3) Put 0 and 20 on this line

4) Could you put numbers on the following line

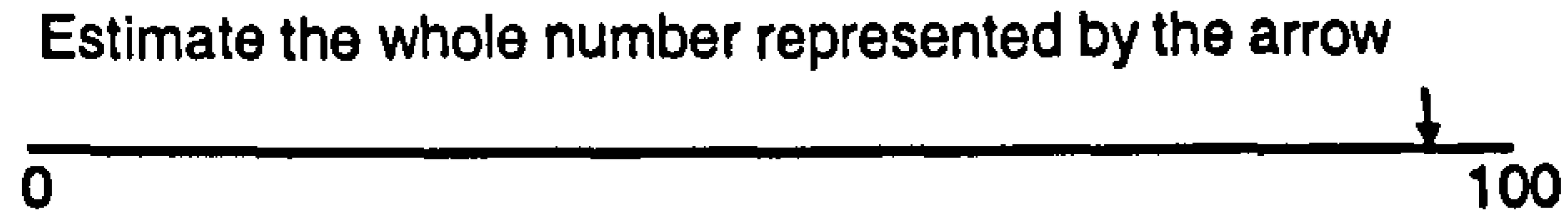
5) Estimate the whole number represented by the arrow



6) Estimate the whole number represented by the arrow

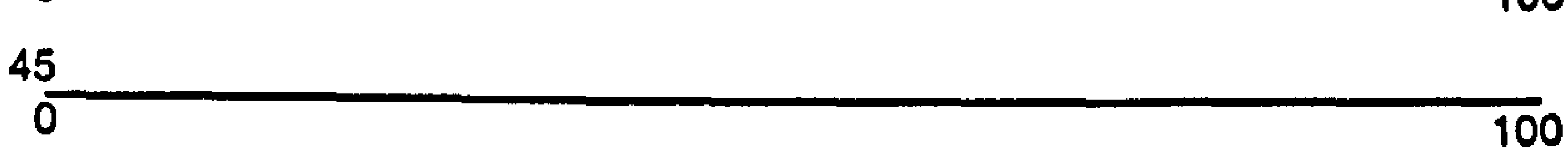


7) Estimate the whole number represented by the arrow



8) Estimate the whole number represented by the arrow

9) Where do you think the following numbers would go on the lines? Mark the point. (The numbers were: 93, 45, 12, 5, 75, 3, 7, 25, 97, 95, 88, 55)



This question carried on in the same way for the rest of the numbers

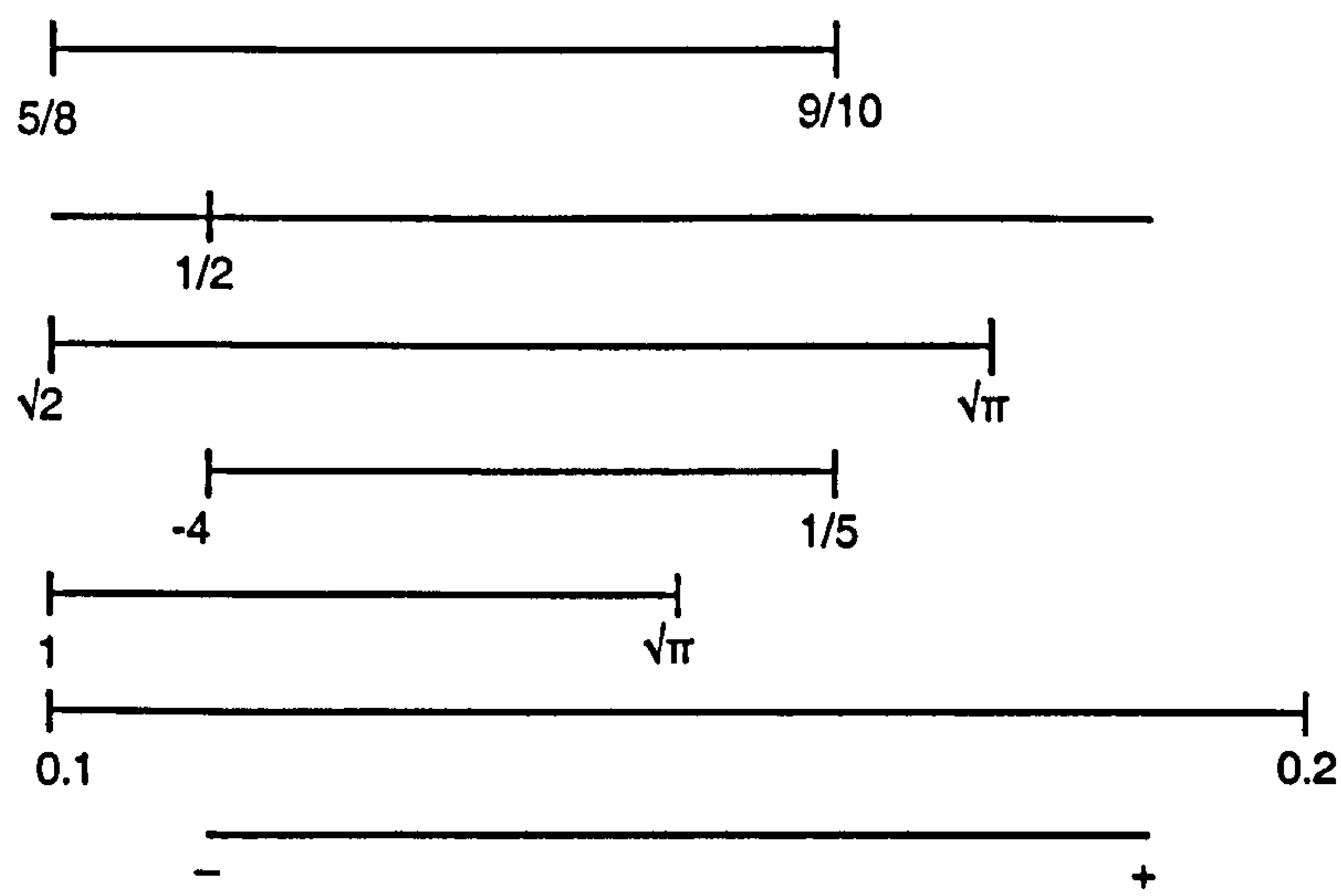
Appendix V

Trainee Teachers Questionnaire

Question

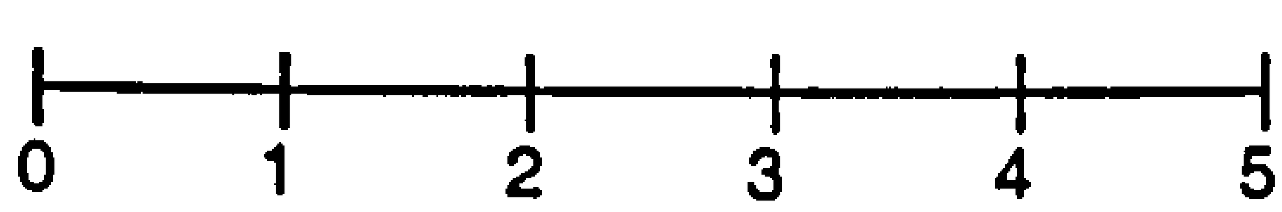
1) How would you define a "number line"?

2) What do you think when you look at these lines? Write down your thoughts.



3) Can you put the previous lines (in question 2) in one line? Explain how you thought.

4) Can you put any more numbers on this line? Write what you think



Appendix VI

Year 2 to Year 6 Data for Question 9

		1	2	3	4	5	6	7	8	9	10	11	12
Numbers to be estimated		3	97	5	95	7	93	12	88	25	75	45	55
Year 6	1	2	96	4	94	6	94	8	86	22	79	38	54
	2	3	95	4	94	6	89	10	87	19	70	39	47
	3	3	93	3	89	12	83	9	75	17	60	36	52
	4	5	93	7	93	14	89	16	85	26	71	43	53
	5	6	90	8	88	10	83	16	81	32	71	45	57
	6	6	93	5	90	10	87	19	80	29	64	38	52
	7	13	81	14	81	23	89	25	81	35	80	43	55
	8	6	86	8	77	11	84	19	64	29	69	37	47
	9	9	87	13	75	36	82	22	64	52	83	45	54
	10	2	85	2	83	7	86	8	76	23	83	34	54
	11	3	91	5	86	8	86	14	71	20	74	36	49
	12	3	90	6	77	10	89	17	68	25	70	44	40
	13	3	89	4	85	8	85	11	75	17	69	32	44
	14	7	74	11	76	21	74	17	48	39	64	29	57
	15	13	89	10	82	19	74	32	64	32	69	55	48
	16	3	93	5	90	5	91	13	79	24	73	44	51
	17	3	89	5	81	9	88	15	67	29	72	39	56
	18	8	87	10	70	15	76	19	60	27	64	41	50
	19	1	95	2	93	3	91	6	87	27	73	42	47
Year 5	1	8	82	12	74	21	65	30	53	39	58	53	41
	2	7	86	7	81	12	83	19	69	27	65	39	55
	3	2	95	4	93	5	91	16	88	25	75	44	53
	4	5	92	8	88	12	79	23	71	27	65	38	59
	5	8	86	9	79	16	76	12	63	29	48	18	51
	6	6	90	9	85	22	85	15	75	29	74	49	52
	7	10	84	12	74	22	71	25	55	35	67	49	28
	8	1	92	2	86	2	78	3	79	8	63	29	51
	9	6	89	11	88	14	86	25	75	45	74	56	54
	10	4	93	8	91	8	89	24	89	39	75	60	71
	11	6	87	10	82	14	81	21	71	36	72	44	51
	12	2	94	12	90	14	86	21	81	19	50	45	57
	13	2	94	5	93	4	87	11	84	12	53	30	51
	14	4	87	6	84	13	79	12	75	25	60	42	55
	15	3	87	9	84	11	86	24	68	27	68	47	53
	16	4	96	5	93	10	91	18	78	23	70	40	51
Year 4	1	3	93	5	90	6	88	7	82	16	72	40	55
	2	7	90	5	80	18	81	12	65	31	76	37	46
	3	4	86	7	81	11	85	14	73	41	64	42	59
	4	12	89	22	86	27	86	38	65	38	68	51	48
	5	6	86	10	70	13	74	25	47	42	67	53	60
	6	16	85	5	88	32	81	61	56	23	60	40	71
	7	4	29	9	24	19	27	88	91	40	90	67	86
	8	11	80		80	14		38	56	54	68	59	57
	9	7	87	13	82	19	81	12	70	35	78	42	52
	10	6	87	4	84	12	88	9	77	18	80	32	55
	11	5	91	7	86	10	85	21	73	29	68	28	50
	12	5	89	7	77	10	81	13	66	27	73	38	48
	13	5	92	5	82	13	85	16	69	30	88	42	54
	14	13	88	16	85	18	82	18	76	28	83	33	41
	15	2	93	3	93	5	89	7	90	22	73	35	49
	16	4	93	8	88	9	87	12	81	27	75	42	53
	17	7	88	11	82	21	76	22	74	31	54	30	61
	18	7	89	12	84	13	85	25	70	30	70	41	56
	19	5	88	7	81	18	87	21	62	32	78	41	54
Year 3	1	4	92	7	83	21	86	8	71	20	62	28	48
	2	10	83	10	67	24	79	25	72	48	65	52	60
	3	5	92	8	65	9	83	18	74	56	62	59	57
	4	7	87	16	82	15	78	37	59	44	54	50	39
	5	7	78	18	87	27	79	40	75	37	78		48
	6	7	18	40	19	19	91	42	82	18	26	40	84
	7	9	89	9	78	25	14	20	79	44	72	31	54
	8	5	88	9	79	12	78	12	83	18	64	37	47
	9	4	92	6	72	10	79	12	73	37	49	45	71
	10	5	91	9	82	12	83	25	73	33	60	47	52
	11	4	88	9	79	12	68	8	64	22	53	18	43
	12	6	88	4	74	24	83	29	66	18	82	54	27
	13	6	90	10	82	12	86	16	77	40	75	48	49
	14	4	92	9	85	19	80	13	71	30	63	33	53
	15	6	91	15	82	17	80	24	74	29	62	41	55
	16	6	45	6	53	10	58	18	28	31	39	28	19
	17	6	93	16	88	12	88	21	79	44	56	42	51
	18	9	88	18	66	36	29	31	87	77	84	59	69
Year 2	1	1	88	2	77	23	84	5	29	77	35	91	53
	2	7	93	17	89	32	87	43	88	47	72	47	42
	3	8	79	15	72	26	78	23	65	37	54	28	54
	4	5	88	8	80	11	84	29	71	27	81	28	45
	5	0	84	0	58	0	50	0	27	0	0	0	0
	6	4	8	2	16	15	4	45	22	30	14	3	30
	8	5	83	8	68	10	80	34	65	61	51	58	69
	9	4	66	6	68	14	71	21	61	32	58	61	43
	10	8	83	19	80	21	91	20	73	37	68	35	47
	11	4	28	7		12	62	42	59	35	80	58	50
	12	3	92	5	91	6	90	13	78	17	75	21	59
	13	5	83	4	76	10	43	25	80	61	41	78	67
	14	4	94	6	90	10	87	13	75	30	93		30
	15	15	95	18	97	31	94	20	90	37	30	43	53
	16	14	65	29	53	31	49	79	72	54	58	78	68

Note: The children are ordered highest achiever to lowest achiever within each year group

Appendix VII

Year 2 to Year 6 Raw Questionnaire Data

	Child	Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8
Year 2	1					96	9	99	5
	2					96	91	99	91
	3					26	11	99	8
	4					80	40	90	30
	5					80	20	90	10
	6					80	50	90	30
	7					80	40	96	20
	8					35	7	18	4
	9					0	6	20	0
	10					40	10	30	5
	11					91	11	97	9
	12					92	11	99	4
	13					35	10	90	4
	14					97	10	99	4
	15					98	9	99	6
	16					30	72	38	4
	17					90	11	99	5
	18					27	61	99	51
	19					92	11	99	7
	20					94	8	99	5
	21					80	40	90	20
	22					90	3	90	2
Year 3	1	a	1/2, 1 1/2, 2 1/2, 3 1/2, 4 1/2	Ends	5,10,15,20,25,30	80	30	90	15
	2	e	Nothing	Ends	2,4,6,8,10,12	98	4	99	2
	3	e	Nothing	Ends	3,6,9,12,15,18	96	7	99	4
	4	e	Nothing	Ends	1,3,5,7,9,11	98	5	99	4
	5	a	Nothing	Ends	0,3,6,9,12,15	90	20	99	2
	6	a	Nothing	Ends	20,30,40,50,60,70	98	20	99	3
	7	a	Nothing	Ends	0,2,4,6,8,10	98	4	99	5
	8	e	Nothing	Ends	1,2,90,1000,10000,99	98	5	99	3
	9	e	Nothing	Ends	2,4,6,8,10,12	98	4	99	2
	10	a	Nothing	Ends	0,10,20,30,40,50	90	20	99	2
	11	a	Nothing	Ends	0,2,4,6,8,10	90	30	99	10
	12	e	Nothing	Ends	0,1,2,3,4,5	97	20	99	10
	13	d	Nothing	Ends	1,5,10,15,25 inside intervals	95	10	99	5
	14	c	Nothing	Ends	0,1,2,3,4,5	80	15	99	5
	15	a	Nothing	0 left end and 20 in middle	0,10,20,30,40,50	90	20	99	2
	16	c	Nothing	Ends	0,1,2,3,4,5	90	3	90	2
	17	e	Nothing	Ends	2,4,6,8,10,12	98	10	99	4
	18	a	Nothing	Ends	12,34,56,78,910 inside inter	23	4	41	2
Year 4	1	a	1/4, 1/2, 1/3, 1 1/2, 1 1/2, 2 1/2, 3 1/2, 4 1/2	Ends	5,15,25,35,45,55	80	30	90	15
	2	c	Nothing	Ends	0,1,2,3,4,5	98	16	99	5
	3	a	3 1/4, 4 1/4	Ends	6,7,8,9,10,11	90	10	99	10
	4	e	0, 0 14, 0 1/2, 0 3/2	Ends & all wholes in between	0,1,2,3,4,5	90	8	99	4
	5	b	Nothing	Ends	0,1,2,3,4,5	19	5	99	3
	6	a	Nothing	Ends & all wholes in between	1,2,3,4,5,6	97	92	99	X
	7	a	Nothing	Ends	0,1,2,3,4,20	99	5	188	3
	8	a	Nothing	Ends	0,1,2,3,4,5	99	3	99	2
	9	a	Makes 6-8 marks within each interval	Ends & all wholes in between with no marks	0,1,2,3,4,5	98	5	99	5
	10	a	1 1/2, 2 1/2, 3 1/2, 4 1/2	0 left end and 20 at quarter of line distance	0,1,2,3,4,5	80	20	95	15
	11	a	Nothing	Ends	0,1,2,3,4,5	90	10	99	5
	12	e	Nothing	Ends	0,1,2,3,4,5	97	5	99	3
	13	a	Nothing	Ends	0,1,2,3,4,5	80	20	98	8
	14	a	Nothing	Ends	20,30,40,50,60,70	98	10	99	3
	15	e	Nothing	Ends	Negative 6 to negative 1	80	30	90	10
	16	a	Nothing	Ends	0,1,2,3,4,5	86	20	95	10
	17	a	0 1/2, 1 1/2, 2 1/2, 3 1/2, 4 1/2	Ends	0,1,2,3,4,5	98	5	99	3
	18	a	0 1/2, 1 1/2, 2 1/2, 3 1/2, 4 1/2	Ends	0,1,2,3,4,5	97	5	99	3
	19	a	Nothing	Ends	1,2,3,4,5,6	98	10	99	5

Year 2 to Year 6 Raw Questionnaire Data (continued)

Year 5	1	e	Misconception 1/1, 1/2, 1/3, 1/4, 1/5	Ends	0,1,2,3,4,5	98	4	99	2
	2	a	Nothing	Ends & all wholes in between	0,1,2,3,4,5	96	9	99	3
	3	d	Nothing	Ends	0,1,2,3,4,5	80	20	93	10
	4	e	1 1/2, 2 1/2, 3 1/2, 4 1/2	Ends & all wholes in between	0,1,2,3,4,5	97	5	99	3
	5	e	1/2, 1 1/2, 2 1/2, 3 1/2, 4 1/2	Ends	0,1,2,3,4,5	97	12	99	7
	6	a	Nothing	0 left end only	Nothing	90	4	6	4
	7	e	Misconception 1/1, 1/2, 1/3, 1/4, 1/5	Ends	0,1/1,1,1/2,2,1/3,4,1/4,5,1/5,	95	1	99	1
	8	a	Nothing	Ends & all wholes in between	0,1,2,3,4,5	80	10	90	10
	9	c	Misconception m0,m1,m2 1/2, m3,m4	Ends & all wholes in between	0,1,2,3,4,5	90	5	99	3
	10	a	Nothing	Ends	5,6,7,8,9,10	80	30	90	3
	11	a	Nothing	Ends & all wholes in between	0,5,10,15,20,25	97	5	99	7
	12	a	1 1/2, 2 1/2, 3 1/2, 4 1/2, 5 1/2	Ends	0,1,2,3,4,5	80	20	90	10
	13	a	Nothing	Ends	Nothing	90	10	98	5
	14	e	Nothing	Ends	0,1,2,3,4,5	95	9	99	5
	15	e	Nothing	Ends & all wholes in between	0,1,2,3,4,5	90	10	99	5
	16	c	1 1/2, 2 1/2, 3 1/2, 4 1/2, 5 1/2	Ends & all wholes in between	0,10,20,30,40,50	94	10	97	3
	17	b	Nothing	Ends & all wholes in between	0,1,2,3,4,5	98	94	99	4
Year 6	1	a	Nothing	Ends	1,2,3,4,5,6	90	30	95	10
	2	d	Quarters & 0.5,1 5,2 5,3 5,4 5	Ends. Marks middle & quarters	0,0 2,0 4,0 6,0 8,1	80	20	95	10
	3	a	Nothing	Ends	0,1,2,3,4,5	97	40	99	20
	4	b	Nothing	Ends	0,1,2,3,4,5	80	20	93	4
	5	b	1/2, 1 1/2, 2 1/2, 3 1/2, 4 1/2	Ends	0,2,4,6,8	97	10	99	5
	6	a	0 1,0 2,0.3 no marks	Ends	1,2,3,4,5,6	90	20	98	7
	7	a	1/2, 1 1/2, 2 1/2, 3 1/2, 4 1/2	Ends	10,8,9,6,5,4	80	25	99	2
	8	a	0 5,1 5,2 5,3 5,4 5 no marks	Ends	0,1,2,3,4,5	97	6	99	3
	9	a	1/2, 1 1/2, 2 1/2, 3 1/2, 4 1/2	Ends	0,1,2,3,4,5	70	25	99	15
	10	a	1/2, 1/1, 2/2, 3/3, 4/4	Ends	0,1,2,3,4,5	96	5	99	4
	11	a	2 5 in middle	Ends	0,1,2,3,4,5	70	30	90	10
	12	a	0 5,1 5,2.5,3 5,4 5	Ends	0,1,2,3,4,5	97	20	99	2
	13	a	0 5,1 5,2.5,3 5,4 5 no marks	0 left end & 20 at quarter	5,10,15,20,25,30	80	20	98	5
	14	a	0.5,1 5,2.5,3 5,4 5	Ends	0,1,2,3,4,5	88	15	99	5
	15	a	Nothing	Ends	7,8,9,10,11,12	97	7	99	3
	16	a	Nothing	Ends	0,1,2,3,4,5	90	10	99	1
	17	a	0.1,0 5,0.9,1.1,1 5,1.9,2.1,2 5,2 9,3 1,3 5,3 9,4 1,	Ends	1,2,3,4,5,6	80	20	95	5
	18	a	0 5,1 5,2 5,3 5,4 5	Ends	0,1,2,3,4,5	98	5	99	1
	19	a	Nothing	Ends	0,1,2,3,4,5	90	20	98	8
	20	d	0 5,1.5,2 5,3.5,4 5	0 left end & 20 at quarter	15,16,17,18,19,20	83	19	93	9

Appendix VIII

The First Step (1892)

The young poet Evmenis
complained one day to Theocritos:
"I've been writing for two years now
and I've composed only one idyll.
It's my single completed work.
I see, sadly, that the ladder
of Poetry is tall, extremely tall;
and from this first step I'm standing on now
I'll never climb any higher."
Theocritos retorted: "Words like that
are improper, blasphemous.
Just to be on the first step
should make you happy and proud.
To have reached this point is no small achievement:
what you've done already is a wonderful thing.
Even this first step
is a long way above the ordinary world.
To stand on this step
you must be in your own right
a member of the city of ideas.
And it's a hard, unusual thing
to be enrolled as a citizen of that city.
Its councils are full of Legislators
no charlatan can fool.
To have reached this point is no small achievement:
what you've done already is a wonderful thing."

Constantine P. Cavafy

Appendix IX

Ithaca (1911)

When you set out on your journey to Ithaca,
pray that the road is long,
full of adventure, full of knowledge.
The Lestrygonians and the Cyclops,
the angry Poseidon -- do not fear them:
You will never find such as these on your path,
if your thoughts remain lofty, if a fine
emotion touches your spirit and your body.
The Lestrygonians and the Cyclops,
the fierce Poseidon you will never encounter,
if you do not carry them within your soul,
if your soul does not set them up before you.

Pray that the road is long.
That the summer mornings are many, when,
with such pleasure, with such joy
you will enter ports seen for the first time;
stop at Phoenician markets,
and purchase fine merchandise,
mother-of-pearl and coral, amber and ebony,
and sensual perfumes of all kinds,
as many sensual perfumes as you can;
visit many Egyptian cities,
to learn and learn from scholars.

Always keep Ithaca in your mind.
To arrive there is your ultimate goal.
But do not hurry the voyage at all.
It is better to let it last for many years;
and to anchor at the island when you are old,
rich with all you have gained on the way,
not expecting that Ithaca will offer you riches.

Ithaca has given you the beautiful voyage.
Without her you would have never set out on the road.
She has nothing more to give you.

And if you find her poor, Ithaca has not deceived you.
Wise as you have become, with so much experience,
you must already have understood what Ithacas mean.

Constantine P. Cavafy